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JAN 80 J T ODEN DAAG29-77-G-0087
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Final Report

APPROXIMATION AND NUMERICAL ANALYSIS OF NONLINEAR EQUATIONS OF EVOLUTION

DAAG-29-77-G-0087

ARO Proposal - 14505-M

Author: J. T. Oden
January 31, 1980

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Important results were obtained in four areas: (1) Existence theorems, approximation theorems, a priori error estimates, numerical schemes, and finally computer codes were developed for the analysis of one-and two-dimensional, one- and two-phase Stefan problems characterized by variational inequalities. (2) Existence theorems, uniqueness theorems, theorems on the stability and asymptotic stability of solutions, and regularity of solutions were developed for a large class of non-

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20. ABSTRACT CONTINUED

linear, convective diffusion problems characterized by pseudo-monotone operators. (3) A priori error estimates for Galerkin and Faedo-Galerkin approximations (defined, in general, by finite element methods) were established for nonlinear convection diffusion problems involving general pseudomonotone operators. (4) Existence theorems were obtained for a large class of nonlinear, degenerate evolution equations with solutions involving free boundaries. Applications to porous media and two-phase Stephan problems were completed.

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APPROXIMATION AND NUMERICAL ANALYSIS OF NONLINEAR

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CHAPTER I

STATEMENT OF THE PROBLEM STUDIED

Many time dependent physical phenomena are characterized by nonlinear parabolic equations, the solution of which is characterized by a sharp front, sometimes a discontinuity, propagating through the solution domain. Among problems of this type are nonlinear convective diffusion problems with dominant convective terms, or Stefan type problems such as the flow of fluids through porous media or the melting and freezing of ice. Such problems are difficult mathematically and numerically because of the poor regularity of the solutions. Moreover, the mathematical theory underlying these problems and their approximations is very much incomplete.

Toward resolving some of these issues, a three-year project was initiated in 1976, designed to study not only the qualitative features of solutions of nonlinear problems of this type, but also in developing numerical schemes for solving such problems. Since then the research has basically taken two somewhat distinct directions. First, a study of the use of variational inequalities as a means of formulating time-dependent Stefan problems was initiated. Classes of problems considered here include the one-phase and two-phase Stefan problems encountered in porous media applications and, in particular, problems of ablation of metals and freezing and thawing of soils. A variety of finite element schemes were developed and studied for these problems, some of which proved to be very effective. Using variational inequalities as a basis, some new numerical methods were developed for two dimensional, two-phase Stefan problems with time dependent boundary conditions. A variety of example problems was solved, and a method was produced which seems to be very effective. Some of the results of this portion of the study have appeared or will appear soon in the literature. The analysis of the one-phase Stefan problem was first completed, including not only the identification of effective numerical schemes but also the development of a priori error estimates for finite element approximations. The studies led to information on the qualitative and quantitative behavior of the solution and its regularity, the behavior of the error, and criteria for the selection of trial functions for finite element approximations. In these studies, Stefan problems were considered without convective terms.

At the end of the first year of the project it became clear that to model realistically certain phenomena characterized by parabolic equations and the propagation of fronts, it would be more appropriate to include convective terms in the formulation. Indeed, the presence of dominant convective terms in convection-diffusion processes is known to lead to solutions with fronts and to notorious numerical difficulties. Toward resolving some of these issues, a theoretical analysis was initiated to study the behavior of highly non-linear parabolic equations which contained convective type terms. Here a study of the theory of evolution problems characterized by pseudo-monotone operators was performed. Existence theorems, uniqueness theorems, regularity theorems, and stability results were derived for operators of the form $A(u) + c|u|^q|\nabla u|^r$, where q and r take on values appropriate to make the operator pseudo-monotone, and A is a non-linear monotone operator. Existence theorems using methods of elliptic regularization were also investigated. Finally, a theory of Faedo-Galerkin approximations and semi-discrete Galerkin approximations was devised and applied to finite element approximations of these equations. A priori error estimates were obtained and guidelines for the development of appropriate numerical methods were established.

In recent months, it was discovered that the qualitative analysis of nonlinear parbolic problems could be substantially generalized to include the effects of degenerate coefficients and to model such complex phenomena as two-temperature heat condition, with degenerate equations and nonlinear convective and diffusion terms, plus the effects of free boundaries. This work has been completed only recently and required considerable effort. Professors Oden, Showalter, and Kikuchi worked on this phase of the project, and the recent work of Showalter on nonlinear evolution equations has proved to be invaluable. We feel that a broad theoretical basis has now been established for further work on approximations and the numerical analysis of this class of problem.

The logical extension of this work will also be the development of numerical algorithms for the study of degenerate nonlinear convection diffusion problems of the type described above and the numerical study of representative two-dimensional problems. Some encouraging preliminary results have already been obtained in this direction.

CHAPTER II

SUMMARY OF THE MOST IMPORTANT RESULTS

Important results were obtained in four areas:

One: Existence theorems, approximation theorems, a priori error estimates, numerical schemes, and finally computer codes were developed for the analysis of one- and two-dimensional, one- and two-phase Stefan problems characterized by variational inequalities.

Two: Existence theorems, uniqueness theorems, theorems on the stability and asymptotic stability of solutions, and regularity of solutions were developed for a large class of non-linear, convective diffusion problems characterized by pseudo-monotone operators.

Three: A priori error estimates for Galerkin and Faedo-Galerkin approximations (defined, in general, by finite element methods) were established for nonlinear convection diffusion problems involving general pseudomonotone operators.

Four: Existence theorems were obtained for a large class of nonlinear, degenerate evolution equations with solutions involving free boundaries. Applications to porous media and two-phase Stephan problems were completed.

CHAPTER III

LIST OF PUBLICATIONS AND REPORTS

(1) Alduncin, G. and J.T. Oden, "Qualitative Analysis and Colloquim Approximat:ons of a Class of Nonlinear Diffusion Problems," SIAM Journal of Mathematical Analysis, (to appear).

(2) Alduncin, G., "Qualitative analysis and Galerkin approximations for nonlinear monotone and pseudomonotone operators in nonlinear diffusion problems", Ph.D. Dissertation, The University of Texas at Austin, Austin, Texas, December 1978.

(3) Alduncin, G., and Oden, J.T., "Qualitative analysis and Galerkin approximations of a class of nonlinear diffusion problems", SIAM Journal of Numerical Analysis (in review).

(4) Alduncin, G., and Oden, J.T., Qualitative analysis and Galerkin approximations of a class of nonlinear diffusion problems, TICOM Report - 78-8, The University of Texas at Austin, Austin, Texas, May 1978.

(5) Ichikawa, Y., "A numerical analysis of a one-phase Stefan problem by variational inequalities", Master's Thesis, The University of Texas at Austin, Austin, Texas, August 1978.

(6) Ichikawa, I., A numerical analysis of a one-phase Stefan problem by variational inequalities, TICOM Report - 78-12, The University of Texas at Austin, Austin, Texas, July 1978.

(7) Ichikawa, I., and Kikuchi, N., "Numerical methods for two-phase Stefan problem by variational inequalities", International Journal of Numerical Methods in Engineering (to appear).

(8) Kikuchi, N., and Ichikawa, Y., "A one-phase multi-dimensional Stefan problem by the method of variational inequalities", International Journal of Numerical Methods in Engineering, (to appear).

(9) Oden, J.T., "Finite element analysis of non-monotone elliptic boundary value problems", U.S. - Japan Seminar on Interdisciplinary Problems in Engineering, Cornell University, Ithaca, New York, August 1978.

(10) Oden, J.T., and Kikuchi, N., "Finite element methods for certain free boundary-value problems in mechanics, Moving Boundary Problems, New York: Academic Press, 1978.

(11) Oden, J.T., Reddy, C.T., and Kikuchi, N., "Qualitative analysis and finite element approximations of a class of nonmonotone nonlinear Dirichlet problems", SIAM Journal of Numerical Analysis (in review).

(12) Oden, J.T., Reddy, C.T., and Kikuchi, N., Qualitative analysis and finite element approximations of a class of nonmonotone nonlinear Dirichlet problems, TICOM Report 78-15, The University of Texas at Austin, Austin, Texas, July 1978.

(13) Showalter, R.E., Nonlinear Diffustion in Heterogeneous Media: Existence Theory and Monotonicity Methods, TICOM Report 79-15, The University of Texas at Austin, October 1979.

CHAPTER IV

SCIENTIFIC PERSONNEL

1. J.T. Oden, Professor of Engineering Mechanics - Principal Investigator

2. N. Kikuchi - Research Engineer/Scientist Associate

3. G. Alduncin - Ph.D. student

4. Y. Ichikawa - Ph.D. student

5. R.E. Showalter, Professor of Mathematics, (devoted a portion of his time to the project during Summer and Fall, 1979).

CHAPTER V

ADVANCED DEGREES EARNED

1. Kikuchi, Noboru, PhD, December 1977.
2. Ichikawa, Yasuaki, MS, August 1978.
3. Alduncin, Gonzalo, PhD, December 1978.

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A One-Phase Multi-Dimensional Stefan Problem by the Method of Variational Inequalities

7. TECHNICAL DISCUSSIONS (APPENDICES)

Appendix A

A One-Phase Multi-Dimensional Stefan Problem by The Method of Variational Inequalities

1. Introduction

Stefan's problem has been considered by many authors since STEFAN [1] formulated in 1891 his mathematical model of the phenomena of a freezing of soils. One- and two-phase Stefan problems have been investigated by KAMENOMONSTSKAJA [1], OLEINIK [1], FRIEDMAN [1], and others. Various numerical procedures have been developed by DOUGLAS and GALLIE [1], JAMET and BONNEROT [1], and others. However, while most existing methods are applicable for one-dimensional problems, not all are extendable to multi-dimensional problems, since many are based on special characteristics of one-dimensional case. We mention here some typical methods.

(1) After discretization with respect to the space variable, the n-th time increment Δt is obtained by the "Stefan" condition,

$$\ell \frac{dL}{dt} = [\![\text{grad } \theta]\!] \quad \text{on the frozen front}$$

so that the following nodal point becomes frozen by the condition

$$\ell \frac{h}{\Delta t} = - [\![\text{grad } \theta^{n-1}]\!]$$

(2) By the Stefan condition, the location L^n of the frozen front is obtained at the time $t = n \Delta t$ through the formula

$$\ell \frac{L^n - L^{n-1}}{\Delta t} = - [\![\text{grad } \theta^{n-1}]\!]$$

and then the domain of ice is discretized by appropriate finite element or difference methods. Here ℓ is the latent heat, L is the position of the frozen front, θ is the temperature field, and $[\![\cdot]\!]$ denotes the difference of the left value and right value on the frozen front.

The first method seems to be possible only for one dimensional problems. The second method is more general, but it becomes difficult for the case in which many disjoint freezing parts exist. In many problems, several frozen fronts may occur simultaneously and one frozen front may grow until it intersects another. Such phenomena are difficult to model on the discretized domain at each time step.

The formulation introduced by DUVAULT [1] enables us to resolve the above difficulties, and has the structure of a strict mathematical analysis. That is, after a special transformation from the temperature field to the freezing index, the problem formulated by the variational inequalities is a problem in which the dependent variable is defined on the whole domain. Moreover, the freezing index $u(\underset{\sim}{x})$ is expected to be continuously differentiable on the whole domain; that is, there is no discontinuity of grad u on the frozen front, while grad θ is discontinuous there. Thus, the problem can be solved within a <u>fixed domain</u> without iteration. The importance of this formulation is that the unfrozen part is identified with the portion where the freezing index remains zero. That is, if we can obtain numerical values of the freezing index, frozen and unfrozen parts can be distinguisehed by the value of u .

The special transformation and the Stefan condition restrict the freezing index u to be non-positive on the whole domain. This leads us to the inequation formulation instead of usual weak forms. This inequation can be solved by appropriate optimization techniques; for example, the projectional S.O.R. method, a penalty method, etc. The formulation by variational inequalities due to DUVAULT is not only powerful for its computational aspects but also well-posed for mathematical and numerical analytic aspects, as shown by LIONS [1], JOHNSON [1], CEA and GLOWINSKI [1], and so on. These numerical analyses are well-established.

In this paper, we describe the method of variational inequalities for one-phase Stefan problems and give a computational technique together with various numerical examples. We compare the results of numerical experiments with TIKHONOV's exact solution of a one-dimensional case [1], confirm our error estimates for finite element methods by numerical experiments, and analyze some nontrivial two-dimensional one-phase Stefan problems.

2. Formulation of One Phase Stefan Problem

2.1. Mathematical Model. Let $D \subset \mathbb{R}^n$ $(n = 1,2,3)$ be an open domain whose subset Ω defines the frozen portion. Let Γ be the boundary of D. On Γ_G negative temperature $g(t)$ is prescribed as a function of time (a Dirichlet boundary condition), and on Γ_c we assume that the temperature θ maintains a value of zero. The flux is prescribed on Γ_F (a Neumann boundary condition). Γ_0 is the interface of ice and water which moves with time t. The function $t = S(x)$ defines the time when the water $x \in D$ changes to ice. That is, $S(x)$ denotes the position of the interface Γ_0 . Its inverse relation, $x = S^{-1}(t)$ is given by $L(t)$. Then a mathematical model of the one phase Stafan problem is formulated as follows (see also Figure 1).

<u>PROBLEM I</u>: For given $g(t)$, $0 \le t \le T$, find $\{S^{-1}(t)\ ,\ \theta(x,t)\}$ such that

$$\frac{\partial\theta}{\partial t} = \nabla \cdot (k\nabla\theta) \quad \text{in} \quad \Omega\ , \tag{2.1}$$

$$\theta = 0 \quad \text{in} \quad D - \Omega\ ,$$

$$\theta(x,t) = g(t) \quad \text{on} \quad \Gamma_G\ , \tag{2.2}$$

$$\theta(x,t) = 0 \quad \text{on} \quad \Gamma_C \tag{2.3}$$

$$\alpha\theta = k\,\frac{\partial\theta}{\partial n} \quad \text{on} \quad \Gamma_F\ , \tag{2.4}$$

$$k\nabla\theta \cdot \nabla S(x) = \ell \quad \text{on} \quad \Gamma_0 \tag{2.5}$$

$$\theta(x,0) = 0 \quad \text{in every} \quad D\ . \tag{2.6}$$

Here $\theta(x,t)$ is the temperature, $k(x)$ the thermal diffusivity, α the constant for the heat radiation, ℓ is defined as $\ell = L\rho C$ where L is the latent heat per unit volume, ρ the density of the material, and C the heat capacity. ■

The solution of the initial boundary value problem (2.1) - (2.6) involves two major difficulties. One is that the domain of ice part is unknown. Another is that the gradient of the temperature θ is not continuous, which makes it difficult to represent the problem variationally.

2.2. Duvaut's Transformation. Let us introduce a special transformation of θ into the freezing index u by

$$u(x,t) = \int_{S(x)}^{t} \theta(x,\tau)\, d\tau \quad \text{in } \Omega \;, \quad u(x,t) = 0 \quad \text{in } D - \Omega \tag{2.7}$$

following DUVAUT [1]. As we mentioned earlier, the new function $u(x,t)$ and its first derivatives $\nabla u(x,t)$ can be shown to be continuous on the whole domain D, while $\nabla\theta(x,t)$ is discontinuous on $\Gamma_o \subset D$. We have

$$\nabla u(x,t) = \int_{S(x)}^{t} \nabla\theta\,(x,\tau)\, d\tau \quad \text{in } \Omega \;, \quad \nabla u(x,t) = 0 \quad \text{in } D - \Omega \tag{2.8}$$

since the temperature θ is zero on Γ_o, i.e., $\theta\big(x, S(x)\big) = 0$. This shows, in fact, that $\nabla u\big(x, S(x)\big) = 0$. Furthermore,

$$\nabla \cdot (k\nabla u) = \int_{S(x)}^{t} \nabla \cdot (k\nabla\theta)\, d\tau - \nabla S(x) \cdot \big[k(x)\nabla\theta\big(x, S(x)\big)\big]$$

Referring to equation (2.1) and (2.5), we have

$$\nabla \cdot (k\nabla u) = \int_{S(x)}^{t} \frac{\partial \theta}{\partial t}\, d\tau - \ell$$

$$= \frac{\partial}{\partial t} \int_{S(x)}^{t} \theta \, d\tau - \ell$$

$$= \frac{\partial u}{\partial t} - \ell \quad .$$

This shows that the field equation in the ice part Ω becomes

$$\frac{\partial u}{\partial t} = \nabla \cdot (k\nabla u) + \ell \quad \text{in} \quad \Omega \, . \tag{2.9}$$

And in the water domain $D - \Omega$ we have

$$u = 0 \quad \text{in} \quad D - \Omega \, . \tag{2.10}$$

It is noteworthy that under the transformation (2.7) the field equation (2.1) does not change its form with the exception of the force term ℓ . Since the temperature θ is below zero degrees centigrade in Ω , the heat potential u is also less than zero:

$$u(x,t) < 0 \quad \text{in} \quad \Omega \, . \tag{2.11}$$

Combining the above considerations, we have

$$\begin{aligned} u\left(\frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - \ell\right) &= 0 \quad \text{in} \quad D \, , \\ u &\leq 0 \quad \text{in} \quad D \, , \\ \frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - \ell &\leq 0 \quad \text{in} \quad D \, . \end{aligned} \tag{2.12}$$

We thus obtain a system (2.12) in which the unknown domain Ω does not appear explicitly.

Thus, the initial boundary problem I is transformed in the following form:

PROBLEM II: For given $\hat{g}(t)$, $0 \le t \le T$, with

$$\hat{g}(0) = 0 \quad , \quad \hat{g}(t) < 0 \quad \text{for} \quad t \in (0,\infty) \,, \tag{2.13}$$

find $u(x,t)$ such that

$$u \left(\frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - \ell \right) = 0 \quad \text{in} \quad D \,, \tag{2.14}$$

$$u \le 0 \quad \text{in} \quad D \,, \tag{2.15}$$

$$\frac{\partial u}{\partial t} - \nabla \cdot (k\nabla u) - \ell \le 0 \quad \text{in} \quad D \,, \tag{2.16}$$

$$u(x,t) = \hat{g}(t) \quad \text{on} \quad \Gamma_G \tag{2.17}$$

$$u(x,t) = 0 \quad \text{on} \quad \Gamma_C \tag{2.18}$$

$$u(x,t) = k \frac{\partial u}{\partial n}(x,t) \quad \text{on} \quad \Gamma_F \tag{2.19}$$

$$u(x,0) = 0 \quad \text{in} \quad D \tag{2.20}$$

REMARK 2.1. In the above formulation, we have assumed that the initial state is saturated by zero degree water, i.e., $\theta(x,0) = 0$ in D . If the initial state is partially ice and partially 0°C water, then it can be modified as follows. Let Ω_0 be the ice domain in D at the initial stage. The heat potential u is now defined by

$$u(x,t) = \int_{S(x)}^{t} \theta(x,\tau)\, d\tau \quad \text{in} \quad \Omega - \Omega_o$$

$$u(x,t) = \int_{0}^{t} \theta(x,\tau)\, d\tau \quad \text{in} \quad \Omega_o \tag{2.7)'}$$

$$u(x,t) = 0 \quad \text{in} \quad D - \Omega$$

This transformation, suggested by FRIEDMAN and KINDERLEHER [1], reduces the field equation to

$$\frac{\partial u}{\partial t} = \nabla \cdot (k\nabla u) + h \tag{2.9)'}$$

where $h = \ell$ in $\Omega - \Omega_o$ and $h = \theta_o(x)$ in Ω_o , $\theta_o(x)$ is the initial temperature in Ω_o . $\blacksquare$

<u>REMARK 2.2</u>. *In the above formulation, we have only considered the boundary* condition

$$k\,\frac{\partial\theta}{\partial n} = \alpha\,\theta \quad \text{on} \quad \Gamma_F$$

If Γ_F is located in the boundary of the domain Ω_o , defined in REMARK 2.1, we may consider the boundary condition

$$k\,\frac{\partial\theta}{\partial n} = \alpha\,\theta + \beta \quad \text{on} \quad \Gamma_F \tag{2.4)'}$$

That is, by integrating from 0 into t in time, the condition

$$k \frac{\partial u}{\partial n} = \alpha u + \beta t \qquad \text{on } \Gamma_F \tag{2.19)'$$

is obtained. The boundary condition (2.19) thus reduced to the condition (2.19)'. We note that if Γ_F is located in the boundary Ω_o and Ω, then (2.19)' cannot be used. ∎

3. Variational Formulation

The corresponding variational formulation for the problem II, defined in the previous section, will be derived in this section. Let $(u,v)_c$ be the "inner product" on the domain C, i.e.,

$$(u,v)_c = \int_c uv \, dx$$

and let $\dot{w} = \partial w/\partial t$.

Suppose that u satisfies (2.13) - (2.20). Let v be arbitrary function such that $v \leq 0$ q.e. in $D \times [0,T]$, $v = 0$ on Γ_c, $v = \hat{g}$ on Γ_G, where T is a positive real number which indicates the time interval of the problem. Then, by integration by parts,

$$(\dot{u}, v-u)_D + \left(k\nabla u, \nabla(v-u)\right)_D - (\ell, v-u)_D$$

$$= \left(\dot{u} - \nabla \cdot (k\nabla u) - \ell, v-u\right)_D + (k\nabla u \cdot n, v-u)_{\partial D}$$

where $n = (n_1, n_2)$ is the outward normal unit vector on the boundary ∂D

of the domain D . From the boundary conditions,

$$(\dot{u}, v-u)_D + \left(k\nabla u, \nabla(v-u)\right)_D - (\ell, v-u)_D$$

$$= \left(\dot{u} - \nabla \cdot (k\nabla u) - \ell, v\right)_D - (\alpha u, v-u)_{\Gamma_F}$$

$$\geq -(\alpha u, v-u)_{\Gamma_F}$$

Here we have used the fact that

$$\left(\dot{u} - \nabla \cdot (k\nabla u) - \ell, v\right)_D \geq 0$$

by (2.12) and $v \leq 0$ a.e. in $D \times [0,T]$. Thus,

$$(\dot{u}, v-u)_D + \left(k\nabla u, \nabla(v-u)\right)_D + (\alpha u, v-u)_{\Gamma_F} \geq (\ell, v-u)_D$$

$$\text{a.e. in } [0,T] \qquad (3.1)$$

is obtained. By integration of (3.1) in time $[0,T]$, we have the variational problem:

<u>PROBLEM III</u>: Find $u \in \tilde{K}$ such that

$$\int_0^T \{(\dot{u}, v-u)_D + \left(k\nabla u, \nabla(v-u)\right)_D + (\alpha u, v-u)_{\Gamma_F}\}\, dt$$

$$\geq \int_0^T (\ell, v-u)_D\, dt \qquad (3.2)$$

for every $v \in \tilde{K}$, with the initial condition $u(x,o) = 0$ a.e. in D , where

$$\tilde{K} = \{v \in L^2(0,T;\ H^1(D)) : \dot{v} \in L^2(0,T;\ L^2(D)) ,$$

$$v = \hat{g} \text{ a.e. on } \Gamma_G \times [0,T] ,\ v = 0 \text{ a.e. on } \tag{3.3}$$

$$\Gamma_c \times [0,T] ,\ \text{and } v \leq 0 \text{ a.e. in } D \times [0,T]\}$$

Here the space $L^2(0,T;\ V)$ means that for every $v \in L^2(0,T;\ V)$

$$\int_0^T \|v\|_V^2\, dt < +\infty$$

where $\|\cdot\|_V$ is the norm of the space V. ■

<u>THEOREM 1</u>. (LIONS [1]), also ICHIKAWA [1]). Suppose that $\text{mes}\,(\Gamma_G) > 0$. Then there exists a unique solution $u \in \tilde{K}$ of the variational problem (3.2) in the set (3.3) such that

$$\dot{u} \in L^2(0,T;\ H^1(D)) \cap L^\infty(0,T;\ L^2(D))$$

$$\dot{u} - \nabla \cdot (k\nabla u) \in L^2(D \times [0,T]) .$$ ■

The regularity for the solution u, i.e., $\dot{u} \in L^\infty(0,T;\ L^2(D))$, enables us to consider the problem:

<u>PROBLEM IV</u>: Find $u \in K$ such that

$$(\dot{u},\ v-u)_D + (k\nabla u,\ \nabla(v-u))_D + (\alpha u,\ v-u)_{\Gamma_F} \geq (\ell,\ v-u)_D$$

$$\text{a.e. in } [0,T] \tag{3.4}$$

for every $v \in K$,

$$K = \{v \in H^1(D): v = \hat{g} \text{ a.e. on } \Gamma_G, \; v = 0 \text{ a.e. on } \Gamma_c, \tag{3.5}$$
$$\text{and } v \le 0 \text{ a.e. in } D\} \quad ■$$

4. Approximation of Variational Inequality

Since it is almost impossible to obtain analytical solutions of PROBLEM III or other equivalent forms except for certain one-dimensional cases, it is natural that we consider approximate methods. Here we describe Galerkin approximations in space and discretization by finite difference schemes in time. We use a finite element scheme for spatial approximations since our problem may have an irregular boundary. It can be shown that the finite element approximation converges to a solution of the given problem.

Let us consider the finite element discretization of PROBLEM IV. Let

$$B(u,v) = (k\nabla u, \nabla v)_D + (\alpha u, v)_{\Gamma_F}$$

$$(u,v) = (u,v)_D, \quad \text{and} \quad (f,v) = (\ell,v)_D$$

Then (3.4) can be written by

$$\left(\frac{\partial u}{\partial t}, v-u\right) + B(u, v-u) \ge (f, v-u) \quad \text{for } v \in K \tag{4.1}$$

Let V be defined by

$$V = \{v \in H^1(D): v|_{\Gamma_G} = \hat{g}(t), \; v|_{\Gamma_c} = 0\}, \tag{4.2}$$

Then the admissible set (3.5) can be written

$$K = \{v \in V: v \le 0 \text{ a.e. in } D\}, \tag{4.3}$$

Now, in general, we want to construct finite element approximations in $H^m(D)$ for some $m \geq 0$. The following arguments are due to ODEN and KIKUCHI [1]. Let P_h be a partition of D into E-subdomains (finite elements) $\{\Omega_e\}_{e=1}^E$ such that

$$\bar{D} = \cup_{e=1}^E \bar{\Omega}_e \quad ,$$
$$\Omega_e \cap \Omega_f = \phi \quad \text{for} \quad e \neq f \tag{4.4}$$

where $\bar{\Omega}_e$ denotes the closure of Ω_e. Let $h_e = \text{dia}\ (\Omega_e)$, and $h = \max\ (h_e)$. We consider a finite dimensional subspace of $H^m(D)$ consisting of polynomials of degree k, $k \geq m \geq 0$. Let S_h be the finite-dimensional subspace of $H^m(D)$ corresponding to each P_h determined by the above polynomial spaces. We construct S_h so that it is an approximation of $H^m(D)$. Let ρ_e = sup {diameters of all spheres in Ω_e}. Suppose we have the condition: there is a constant $C_c > 0$ such that

$$h_e/\rho_e \leq C_o \quad \text{for every} \quad e\ , \quad 1 \leq e \leq E \quad ,$$

then we assume there exists a constant $C > 0$, which is independent of u and h, such that

$$\|u - \pi_h u\|_{H^s} \leq C h^\sigma \|u\|_{H^r}$$
$$r \geq 0 \quad , \quad 0 \leq s \leq \min\{m,r\} \quad , \tag{4.5}$$
$$\sigma = \min\{k+1-s,\ r-s\} \quad ,$$

for every $u \in H^r(D)$. Note that $\pi_h\colon H^r(D) \to H^s(D)$ is a projection of $H^r(D)$ onto $S_h(D) \subset H^m(D)$. Then the family $\{S_h\}$ can provide a basis of the real Hilbert space for

$$\bigcup_{0<h\le h_{max}} S_h(D) \quad \text{is everywhere dense in} \quad H^m(D) \tag{4.6}$$

Let $\{K_h\}$ be a closed convex subset of S_h ($K_h \subset S_h$) which has the following properties: for all $v \in K \subset V$, a sequence $\{v_h\}$ in K_h can be constructed as

$$v_h \to v \in K \quad \text{strongly for} \quad h \to 0 \ ,$$

and the weak limit u of the sequence $\{u_h\}$ in K_h also belongs to K. Note that in general $K_h \neq K$.

Now in our one phase Stephan problem, $m = 1$ (recall (4.2)), and $k = 1$ is taken. Thus, we select piecewise linear polynomials $\{\phi_i\}$ as a basis of an N-dimensional space $S_h(D)$. Note that in (4.5), $r = k+1 = 2$. Thus, K_h can be defined by

$$K_h = \{\underset{\sim}{v} \in S_h : \underset{\sim}{v} = \sum_{i=1}^{N} v_i \ \phi_i \ (v_i \in \mathbb{R}) \ , \ \underset{\sim}{v}|_{\Sigma^G} = \hat{g}|_{\Sigma^G} \ ,$$

$$\underset{\sim}{v}|_{\Sigma^C} = 0 \ , \ \underset{\sim}{v}|_{\Sigma} \le 0\} \tag{4.7}$$

where Σ^G, Σ^C, Σ denote the sets of all nodal points on Γ_G, Γ_C and in D respectively. Note that $\underset{\sim}{v}|_{\Sigma} \le 0$ implies $v_i \le 0$ for all i, $1 \le i \le N$. Then $K_h \subset K$. For simplicity, we denote

$$u \simeq \underset{\sim}{u} = \sum_{i=1}^{N} u_i \ \phi_i = u_i \ \phi_i \tag{4.8}$$

where $u \in K$, $\underset{\sim}{u} \in K_h$.

Discretization in Space. Let $\underset{\sim}{u} = u_i\phi_i$, $\underset{\sim}{v} = v_i\phi_i$. Substituting these into (4.1), we have

$$M_{ij} \frac{\partial u_i}{\partial t} (v_j - u_j) + K_{ij} u_i (v_j - u_j) - f_j (v_j - u_j) \geq 0 \tag{4.9}$$

where M_{ij} is a mass matrix, K_{ij} a stiffness matrix, f_j a force vector given by

$$\begin{aligned} M_{ij} &= (\phi_i , \phi_j) , \\ K_{ij} &= B(\phi_i , \phi_j) , \\ f_j &= (f , \phi_j) . \end{aligned} \tag{4.10}$$

We know that M_{ij} and K_{ij} are symmetric by (4.10). Then, interchanging i and j ,

$$M_{ij} \frac{\partial u_j}{\partial t} (v_i - u_i) + K_{ij} u_j (v_i - u_i) - f_i (v_i - u_i) \geq 0 \tag{4.11}$$

Discretization in Time. For the time direction, we apply a finite difference scheme:

$$\begin{aligned} M_{ij} \frac{u_j^{n+1} - u_j^n}{\Delta t} (v_i - u_i) &+ \Big[\theta\, K_{ij}\, u_j^{n+1} \\ &+ (1-\theta)\, K_{ij}\, u_j^n\Big] (v_i - u_i) - f_i (v_i - u_i) \geq 0 , \end{aligned} \tag{4.12}$$

where u_j^n denotes j-th point value of $u = u_i \phi_i \in K_h$ at the time step n , Δt the time difference between n+1 and n time step, θ , $0 \leq \theta \leq 1$ is the Crank-Nicolson coefficient. For $\theta = 1$, (4.12) becomes

$$(\partial u_h, v_h - u_h) + B(u_h, v_h - u_h) - (f, v_h - u_h) \geq 0$$

where ∂ denotes the implicit finite difference operator such that

$$\partial u_h^{n+1} = \frac{u_h^{n+1} - u_h^n}{\Delta t} . \qquad (4.13)$$

Inequality (4.12) can be written explicitly in terms of u_j^{n+1} as follows:

$$\hat{K}_{ij} u_j^{n+1} (v_i - u_i^{n+1}) \geq \hat{f}_i (v_i - u_i^{n+1}) , \qquad (4.14)$$

where

$$\hat{K}_{ij} = M_{ij}/\Delta t + \theta K_{ij} ,$$

$$\hat{F}_{ij} = M_{ij}/\Delta t - (1 - \theta) K_{ij} ,$$

$$\hat{f}_i = f_i - \hat{F}_{ij} u_j^n .$$

For the implicit finite difference scheme $(\theta=1)$ in time and the linear finite element method in space, the following error estimate has been established by JOHNSON [1], and also by ODEN and KIKUCHI [1].

THEOREM 2. (Error Estimate). Let u be the solution of (3.4) and (3.5). Let u_h^n be the solution of the discrete problem (4.12) and (4.7) at the n-th step in time ($\theta = 1.0$), and let

$$e^n = u - u_h^n$$

Suppose that the speed of propagation of the frozen front is of order t^α. Suppose that $u \in L^\infty(0,T: H^2(D))$, $\dot{u} \in L^2(0,T; H^1(D))$, and $\dot{u} - \nabla \cdot (k\nabla u) - \ell \in L^\infty(0,T: L^\infty(D))$. Then

$$\underset{n}{\text{Max}} \; \|e^n\|_0^2 + \alpha \, \|e^{n+1}\|_1^2 \, \Delta t \;\leq\; C(h^2 + \Delta t^\mu)$$

$$\mu = \min(2, 3 + 2\alpha)$$

where α, c are constants independent of u and u_h^n. ∎

4. Methods of Optimization

The discrete problem (4.14) on the closed convex set (4.7), defined in the section 3.2, can be solved by the projectional pointwise S.O.R. method as long as the matrix $\hat{K}_{ij}$, (4.14), is positive definite, c.f. CEA and GLOWINSKI [1].

Projectional S.O.R. Method

(i) Pick up $u_h^{n+1}(0) \in K_h$; for example, set $u_h^{(0)} = 0$

(ii) Suppose that k-th iteration $u_h^{n+1}(k) \in K_h$ is known.

$$U_{h,i}^{n+1}\,(k+0.5) \;=\; (1-\omega)\,U_{h,i}^{n+1}\,(k) + \omega\left(-\sum_{j=1}^{i-1}\hat{K}_{ij}\,U_{h,j}^{n+1}\,(k+1) - \sum_{j=i+1}^{N}\hat{K}_{ij}\,U_{h,j}^{n+1}\,(k) + \hat{f}_i\right)\Big/\hat{K}_{(ii)} \tag{4.15}$$

(iii) $U_{h,i}^{n+1}\,(k+1) \;=\; \mathrm{Min}\left(0,\, U_{h,i}^{n+1}\,(k+0.5)\right)$

The iteration factor ω is chosen so that $0 < \omega < 2$. Its optimal value is decided by numerical experiments, while the convergence of the above algorithm is obtained for $0 < \omega < 2$, if $\hat{K}$ is positive definite.

The convergence of the scheme (4.15) is understood by the following criterion:

$$\text{tolerance} = \frac{\sum_{i=1}^{N}\left|U_{h,i}^{n+1}\,(k+1) - U_{h,i}^{n+1}\,(k)\right|}{\sum_{i=1}^{N}\left|U_{k,i}^{n+1}\,(k+1)\right|} \leq \varepsilon_c \tag{4.16}$$

where ε_c is a positive small number.

It is certainly true that there are several other ways to solve numerically the problem of variational inequality (4.14). For example, the penalty method, the Lagrange multiplier method, the fixed point method, and so on. Here we merely mention the final forms of the above methods which we employed. Some numerical results of a one dimensional problem using several optimization methods are shown in the following section (see Example 5.5).

Fixed point method: Instead of $(4.15)_2$, we have

$$U^{n+1}_{k,i}\,(k+0.5) = U^{n+1}_{k,i}\,(k) - \rho\left(\sum_{j=1}^{i-1}\hat{K}_{ij}\,U^{n+1}_{k,j}\,(k+1) + \sum_{j=i+1}^{N}\hat{K}_{ij}\,U^{n+1}_{k,j}\,(k) - \hat{f}_i\right) \tag*{$(4.15)_F$}$$

The step (iii) in (4.15) is also applied. Here ρ , $0 < \rho < 1$, is a contraction factor which strongly depends on the problem and the discretization.

Lagrange multiplier method: We employ this method together with the concept of the iteration scheme, i.e.,

$$U^{n+1}_{k,i}\,(k+1) = (1-\omega)\,U^{n+1}_{k,i}\,(k) + \omega\left(-\sum_{j-1}^{i-1}\hat{K}_{ij}\,U^{n+1}_{k,i}\,(k+1) - \sum_{j=i+1}^{N}\hat{K}_{ij}\,U^{n+1}_{k,i}\,(k) + \hat{f}_i + q_i(k+1)\right)\Big/\hat{K}_{(ii)} \tag*{$(4.15)_L$}$$

Here we have Lagrange multipliers q_i defined by

$$q_i(k+1) = \sum_{\ell=1}^{k} q_i(\ell) - \lambda \operatorname{Min} \left(0,\ U_{k,i}^{n+1}(k)\right)$$

where λ is the iteration factor. The value of λ also depends on the problem. Note that ω, $0 < \omega < 2$, is the iteration factor, and we do not need to have the step (iii) in (4.15).

<u>Penalty method</u>: This is a direct application of the penalized equation of (3.1) defined by

$$\dot{u}_\varepsilon - \nabla \cdot (k \nabla u_\varepsilon) + u_\varepsilon^+/\varepsilon = f \quad \text{a.e. in } D \times [0,T]$$

where $u_\varepsilon^+ = \operatorname{Sup}(0,\ u_\varepsilon)$, and ε is a small enough constant. For details, see LIONS [1]. Then an iteration scheme can be constructed by the following form:

$$U_{k,i}^{n+1}(k+1) = (1-\omega)\, U_{k,i}^{n+1}(k) + \omega \left(-\sum_{j=1}^{i-1} \hat{K}_{ij}\, U_{k,i}^{n+1}(k+1) \right.$$

$$\left. - \sum_{j=i+1}^{N} \hat{K}_{ij}\, U_{k,i}^{n+1}(k) + \hat{f}_i \right) \Big/ \bar{K}_{(ii)} \qquad (4.15)_P$$

where $\bar{K}_{(ii)} = \hat{K}_{(ii)} + \operatorname{Max}(0,\ S_i)/\varepsilon$,

$$S_i = U_{k,i}^{n+1}(k) \Big/ \left| U_{k,i}^{n+1}(k) \right| .$$

5. Numerical Examples

Our numerical scheme obtained in (4.15) is used for fixed mesh of finite elements and arbitrary time interval (for the stability of the numerical scheme, it is required to be small enough if $0 \le \theta < 1/2$ is used in (4.12)), while other methods, for example, JAMET and BONNEROT [1], DOUGLAS and CALLIE [1], employ variable meshes at each time step, or variable time interval Δt for the fixed mesh to obtain the position of the frozen front using the Stefan condition. The latter method can be applied only for one dimensional problems. Why we can use the fixed mesh is that the freezing index u and its derivative grad u is continuous on the whole domain including the region of ice, water and the frozen front, as discussed in Section 2. Thus we can construct the variational form and its approximation without any restriction of dimension of space.

5.1. One Dimensional Case. We have to check the validity of the formulation of variational inequalities compared with some exact solutions, since for one dimensional problem analytical solutions are known, see for example, TIKHONOV [1].

Suppose that the following one dimensional problem is considered.

$$\frac{\partial \theta}{\partial t} = \kappa^2 \frac{\partial^2 \theta}{\partial x^2} \quad , \quad x \in [0,1]$$

$$\theta(x,0) = 0 \quad \forall\ x \in [0,1] \quad ; \text{ initial condition} \qquad (5.1)$$

$$\theta(0,t) = -1 \quad \forall\ t \in \mathbb{R}_+ \quad ; \text{ boundary condition}$$

$$\kappa \nabla \theta \cdot \nabla S(x) = \ell \quad \text{on } \Gamma_0 \quad ; \text{ Stefan condition}$$

The thermal diffusivity k is given by $k = \kappa^2$. Then following TIKHONOV [1], the exact solution is obtained by

$$\theta(x,t) = \begin{cases} -1 + C\,\Phi\left(x/2\kappa\sqrt{t}\right) , & \text{for } x \leq \alpha\sqrt{t} \\ 0 , & \text{for } x > \alpha\sqrt{t} \end{cases} \tag{5.2}$$

where $\Phi(x)$ is the error function, C the constant given by $C = \Phi(\alpha/2\kappa)^{-1}$, α the constant which determines the frozen front by $x = \alpha\sqrt{t}$. Note that α is obtained by solving a transcendental equation.

The exact position of the frozen front is not obtained exactly, but is obtained approximately by the freezing index.

It is notable that the formulation by variational inequalities of one phase problem does not require the homogeneity of the material constants k and ℓ , while two phase problem does, see KIKUCHI and ICHIKAWA [1].

Example 5.1. Let us select $k = 1.0$ and $\ell = 1.0$ $\forall\, x \in [0,1]$. We use linear finite elements with the mesh size $h = 0.1$. The time interval Δt is 0.1 uniformly. Then at time $t = 0.2$, the numerical results are compared with the Tikhonov's solution in Figure 5.1, which gives good agreement. Here the temperature θ_j^n at j-th nodal point on n-th time step is approximated as $\theta_j^n = (u_j^n - u_j^{n-1})/\Delta t$. Since the linear finite element is used, the gradient ∇u_i^n in i-th element is constant in each element and is obtained by $(u_j^n - u_{j-1}^n)/h$. Numerical values of the gradient u_i^n are corresponding with the Tikhonov's solution at the center of each element exactly. ∎

Example 5.2. In the previous example, ten finite elements have been used for the discretization, which is too coarse in order to get the position of the frozen front properly, while the temperature and the gradient of the freezing index could be obtained closely enough to the exact values. Thus, we compute the same model with fine mesh ($h = 0.01$, i.e., 100 elements), and obtain more precise position of the frozen front. These results are shown in Figure 5.2. ■

Example 5.3. The case of non-homogeneous domain, i.e., $k_1 = 1.0$ for $x \in [0,0.3)$ and $k_2 = 100$ for $x \in (0.3,1]$, is considered. Figure 5.3 designates the difference of the propagation speed of the frozen front between the homogeneous and the non-homogeneous region. Since $k_1 << k_2$, the front propagates more rapidly in $(0.3,1]$ for the non-homogeneous case. Even though the values of k_1 and k_2 are very different, the projectional S.O.R. method converges within almost the same number of iterations as the case of uniform material domain. ■

5.2. Error Norms. Since the exact solution is known for one dimensional problem, we can compute discrete error norms, and observe the correspondence of its theoretical estimate given in THEOREM 2.

Example 5.4. The model problem is the same as one given in Example 5.1, i.e., with uniform $k = \ell = 1.0$. In Figure 5.4, the results of computations of error norms are shown. Results indicate that the order of the error in H^1-norm is exactly h which agrees with the theoretical estimates and that the order of the error in L^2-norm is $h^{1.6}$. ■

5.3. Comparison of Several Optimization Schemes. As mentioned in the previous section, there are several methods to solve the optimization problem (4.14). Here we compare numerical results of those methods for a one dimensional steady state problem.

Suppose the following one dimensional steady state problem:

$$u \in K: \quad (u', (v-u)') \geq (\ell, v-u) \quad \forall\, v \in K \tag{5.1}$$

where $K = \{v \in H^1(0,1): v(x) \leq 0 \text{ a.e. in } [0,1],\ v(0) = 0.25\}$, and we choose $\ell = 1$. Its exact solution is given by

$$u(x) = \begin{cases} -\frac{1}{2}\left(x - \frac{1}{\sqrt{2}}\right)^2, & \text{if } 0 \leq x \leq \frac{1}{\sqrt{2}} \\ 0, & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

Example 5.5. The domain $[0,1]$ is divided into 20 finite elements. Using the numerical schemes obtained in Section 4, the optimal values of ω, λ, ρ, etc. for each method are obtained as follows:

Projective S.O.R.	$\omega = 1.6$	
Lagrange Multiplier	$\lambda = 0.04$	(using $\omega = 1.0$)
Fixed Point	$\rho = 0.04$.	

It is notable that each optimal values strongly depend upon the problem itself. However, λ and ρ may be chosen by the following criterion:

$$\lambda, \rho \;=\; C \cdot \mathrm{Max}(U_i/f_i)$$

where $C = 0.01 - 0.05$, U_i and f_i the generalized displacement and force, respectively, at a certain point.

For the penalty method the parameter ε has to be chosen small enough in order to get more accurate results, but it depends on its discrete element length h , too. The selecting criterion of ε may be

$$\varepsilon \;=\; C \cdot h$$

where h is the mesh size (for any space dimension) and $C = 10^{-3} - 10^{-4}$.

Table 5.1 exhibits the comparison of the results by these methods via the exact solution. We note that the projective S.O.R., the fixed point and the penalty methods are controlling the generalized displacement directly, while the Lagrange multiplier method is controlling the generalized force. According to the results, only the Lagrange multiplier method does not give the satisfactory result. Even though we iterate 400 times, the tolerance is bigger than 1.0E-5 in the Lagrange multiplier method.

The projective S.O.R. method is most effective among them in this problem.

The converging rate $O(\varepsilon)$ of the penalty method in L^2-norm as $\varepsilon \to 0$ is almost 1.0, as shown in Figure 5.5. However, for $\varepsilon < 10^{-2}$, a small rate is obtained, which depends on the round off error of finite element discretization. The number of iterations for convergence of (4.15) are almost the same (about 40 times), that is, it does depend on ε , if $\varepsilon < 1$.

5.3. Two Dimensional Case. Since the formulation and the numerical procedure described earlier are independent of the dimension of the problem, two dimensional problems can be solved without special considerations. Here a two dimensional model is examined for Δt , ω , and θ in (4.14) and (4.15). Then the effect of lamping of the mass matrix M_{ij} is discussed. For the optimization, the projective S.O.R. method is employed in the following examples since it is most efficient as shown in Example 5.5.

Example 5.6. The numerical model is shown in Figure 5.6(a). This model is selected because the two frozen fronts become coupled after some finite time, so that any other methods might have difficulties to solve this problem. Suppose that $k = 1.0$, $\ell = 1.0$ and $h = 1.0$, i.e., the number of finite elements is $10 \times 10 = 100$. The maximum tolerance ϵ_c given in (4.16) is 1.0E-5 . Let us fix the above dimensions in the following examples.

For

$$\Delta t = 0.5 ,$$

$$\omega = 1.0 , \quad \text{for the projective S.O.R.}$$

$$\theta = 1.0 \quad \text{for time discretization}$$

the numerical results are shown in Figure 5.6(b). ■

Example 5.7. Figure 5.7 exhibits the case $\Delta t = 0.1$ at time $t = 5.0$, which gives us almost the same results as the case $\Delta t = 0.5$ as shown in Figure 5.6(b). Here $\theta = 1.0$ and $\omega = 1.0$ are used. According to the results, the time increment Δt may not influence the numerical results so much, if $\theta = 1.0$ (i.e., implicit scheme) is chosen.

Example 5.8. Under the same conditions as in Example 5.6 except the relaxation factor ω, its affection to convergence is checked. We note that ω should be selected in the range such as $0 < \omega < 2$. For $\omega = 1.0$, 1.4, and 1.8, the number of iterations is given in Table 5.2 for each ω. The case of $\omega = 1.0$, i.e., the projective "Guass-Seidel" method, gives the fastest convergence. The calculated temperature field θ is the same for any case of ω.

Example 5.9. Stability of the numerical scheme (4.14) is checked here for some Crank-Nicolson's θ. Since the matrices M_{ij} and K_{ij} are constant for each time step, the results of the linear parabolic problem may be applied; the characteristic equation of (4.14) is written as

$$\det P^T[(I+\theta\,\Delta t\, M^{-1}K)\lambda + \left(-I+(1-\theta)\Delta t\, M^{-1}K\right)]\, P = 0 \tag{5.3}$$

where $M = M_{ij}$, $K = K_{ij}$, $I = \delta_{ij}$ and P is the orthogonal transformation associated with $M^{-1}K$, i.e., $P^T M^{-1}K\, P$ reduces to eigenvalues $m_1 > m_2 > \ldots > m_N$. Then λ in (5.3) is obtained by

$$\lambda_i = 1 - m_i/(1+\theta\,\Delta t\, m_i)$$

for each m_i. In order that the scheme (4.14) is stable, it is required that $\mathrm{Max}\,|\lambda_i| \leq 1$. Clearly $\lambda_i < 1$, so that we must have

$$(1-2\theta)\Delta t\, m_1 \leq 2 \tag{5.4}$$

Thus if θ is selected between 1/2 and 1, the scheme is unconditionally stable. However, for $\theta < 1/2$ the scheme may become unstable. In fact,

$\theta = 0.25$ and $\Delta t = 0.5$ give an unstable example as shown in Figure 5.8.

Example 5.10. The effect of lamping of the mass matrix M_{ij} is discussed. Let $\bar{M}_{ij}$ be a lamped mass matrix of the consistent mass matrix M_{ij} defined by

$$\bar{M}_{ij} = \begin{cases} \sum_{j=1}^{N} M_{ij} & \text{if } i = j , \\ 0 & \text{if } i \neq j . \end{cases} \tag{5.5}$$

That is, the non-diagonal terms are added up to its diagonal. It is notable that some singular frozen fronts at the first time step $t = 0.5$ are observed as shown in Figure 5.9(a), which is considered to be a discretization error. However, such a singular behavior on the frozen front is not observed if the lamped mass scheme is applied as shown in Figure 5.9(b). Furthermore, after several time steps are passed, the temperature field becomes entirely the same in both cases as shown in Figure 5.10. The lamped mass scheme gives a kind of "smoothing" effect to the solution.

The diagonal terms $\bar{M}_{ii}$ of the lamped mass matrix are always greater than the diagonal terms M_{ii} of the consistent mass matrix (where i is not summed); the procedure of lamping always gives more stability than consistent scheme for $0 \leq \theta < 1/2$.

6. Conclusion

We have shown the theory and applications of one phase Stefan problems using the freezing index. Following DUVAULT [1], the problem described by the temperature field has been transformed to the variational inequality in terms of the freezing index. Applying finite element methods in space and finite difference methods in time, the variational inequality has been discretized into a system of linear inequalities, which can be solved by optimization methods. In this article, four kinds of methods: the projectional S.O.R. method, the projectional fixed point method, the Lagrange multiplier method, and the penalty method have been introduced and carefully examined for their speed of convergence using a stationary problem. Along our numerical experiments, the projectional S.O.R. method is the fastest optimization method among them.

Using the Tikhonov's one dimensional solution, the numerical results by the variatonal inequality have been compared, and very close agreement has been obtained. Moreover, using the same one dimensional example, the convergence of finite element methods has been checked numerically for the case of linear interpolations. Numerical results have agreed with the theoretical one, again.

Several nontrivial two dimensional problems have been performed, and the choice of Δt, θ, and ω have been discussed numerically. Furthermore, the effect of lumping of the mass matrix has been checked. The Crank-Nicolson's θ for time integration should be selected between 1/2 and 1. According to numerical experiments, the iteration factor of the projectional S.O.R. method $\omega = 1.0$ is recommended. Lumping of the mass matrix gives smooth frozen fronts at the first few steps of time integration.

Thus, the formulation of one phase Stefan problems by the freezing index is fairly effective for multi-dimensional problems as shown in our discussions.

References

[1] CEA, J. and GLOWINSKI, R., "Sur des méthodes d'optimisation par relaxation," Revue Francaise d'automatique, informatique, recherche operationnelle, R3, 1973, p. 5-32.

[1] DOUGLAS, J. and GALLIE, T.M. "On the Numerical Integration of a Parabolic Differential Equation Subject to a Moving Boundary Condition," Duke Math. J., 22, 1955, 557-571.

[1] DUVAULT, G., "Equation aux Derivees Partielles - Resolution d'un Problème de Stefan (Fusion d'un Bloc de Glace à Zéro Degré)," C.R. Acad. Paris, 1973, p. 1461-1463.

[1] FRIEDMAN, A., "The Stefan Problem in Several Space Variables," Trans. Amer. Math. Soc., 132, 1968, p. 51-87.

[1] FRIEDMAN, A. and KINDERLEHRER, D., "A One Phase Stefan Problem," Ind. Univ. Math. J., 24, 11, 1975, p. 1005-1035.

[1] ICHIKAWA, Y., "A Numerical Analysis of a One Phase Stefan Problem by Variational Inequalities," M.S. Thesis, University of Texas at Austin, 1978.

[1] JAMET, P. and BONNEROT, R., "Numerical Computation of the Free Boundary for the Two-Dimensional Stefan Problem by Finite Elements," J. Compt. Phys.,

[1] JOHNSON, C., "A Convergence Estimate for an Approximation of.a Parabolic Variational Inequalities," SIAM J. Numer. Anal., Vol. B, No. 4, 1976, p. 599-603.

[1] KAMENOMONSTSKAJA, N. and ICHIKAWA, Y., "On the Stafan Problem," Math. Sb., 53, 1961, p. 489-514.

[1] KIKUCHI, N. and ICHIKAWA, Y., "Numerical Methods for Two Phase Stefan Problems by Variational Inequalities," to appear.

[1] LIONS, J.L., Sur Quelques Questions d'Analyse de Mecanique et de Control Optimal, Les Press de Universite de Montreal, 1976.

[1] ODEN, J.T. and KIKUCHI, N., "Finite Element Methods for Certain Free Boundary-Value Problems in Mechanics," Moving Boundary Problems, ed. by D.G. Wilson, A.D. Solomon and P.T. Boggs, Academic Press, 1978.

[1] OLEINIK, O.A., "A Method of Solution of the General Stefan Problem," Soviet Math. Dokl., 1, 1960, p. 1350-1354.

[1] STEFAN, J., "Über die Theorie der Eisbildung, Insbesondere über die Eisbildung in Polarmeere," Ann. Phys. Chem., 42, 1981, p. 269-286.

[1] TIKHONOV, A.N., Equations of Mathematical Physics, Pergamon Press, Oxford, 1963, p. 283-288.

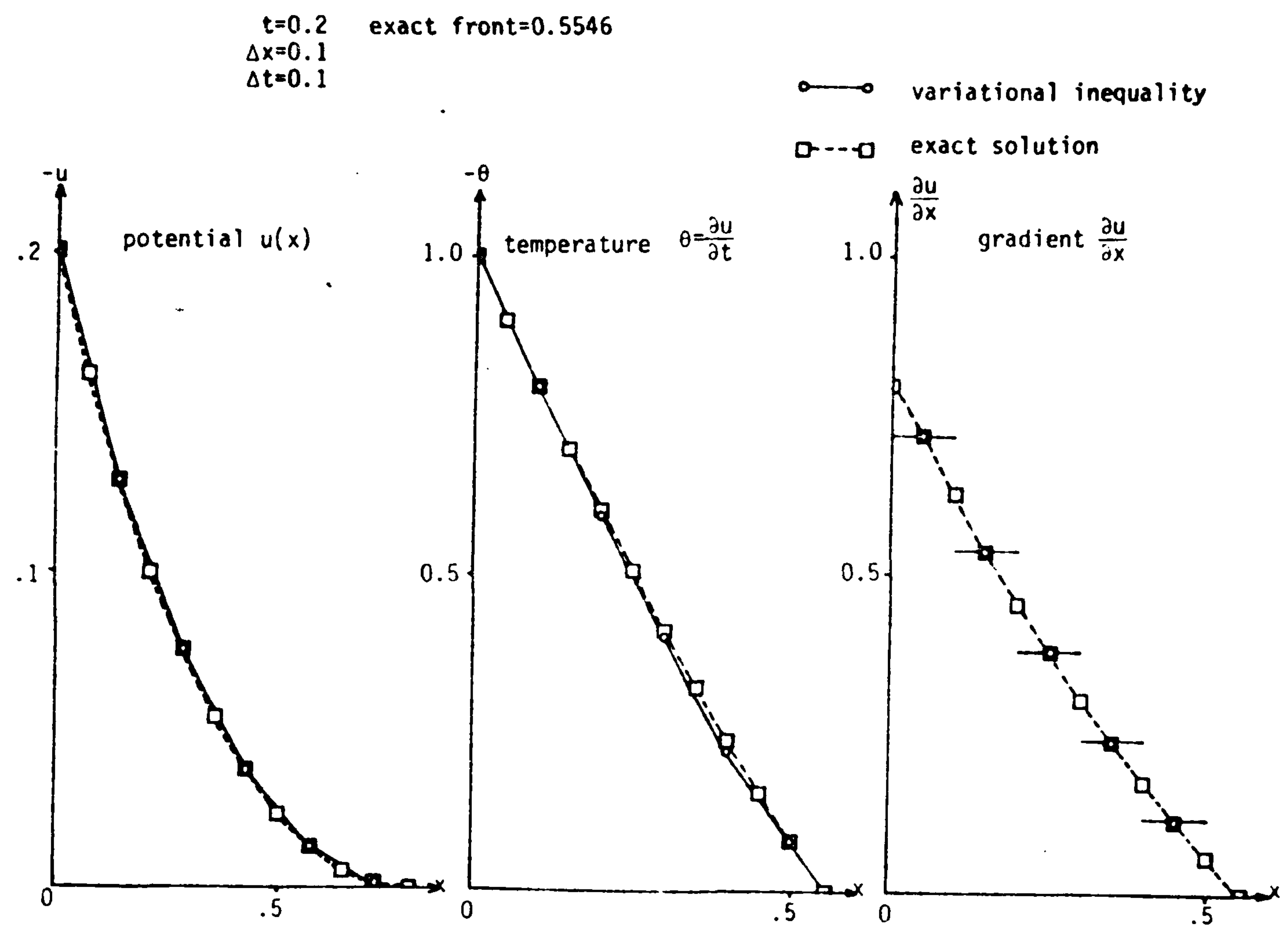


Figure 5.1 Comparison of variational inequality and exact solution (one dimension)

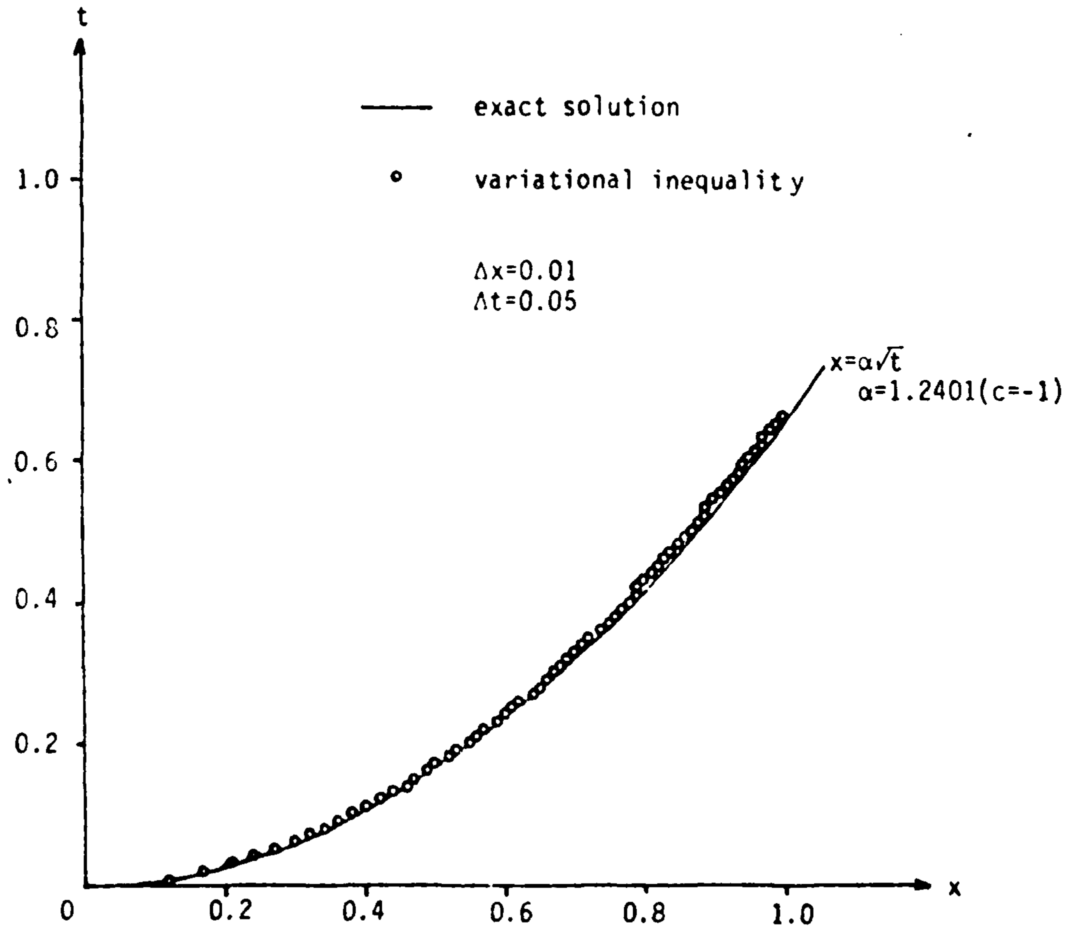


Figure 5.2 Front propagation in one-dimensional model

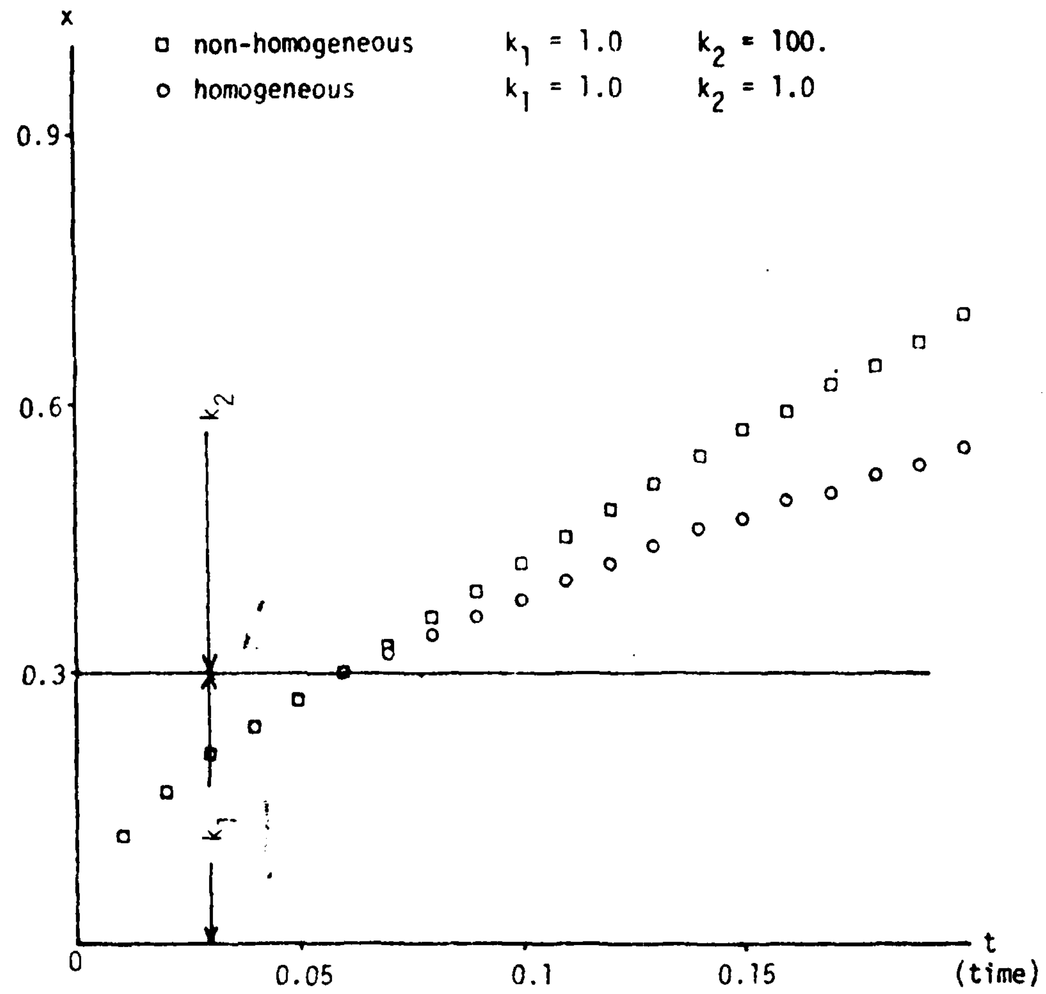


Figure 5.3 Front propagation in non-homogeneous domain

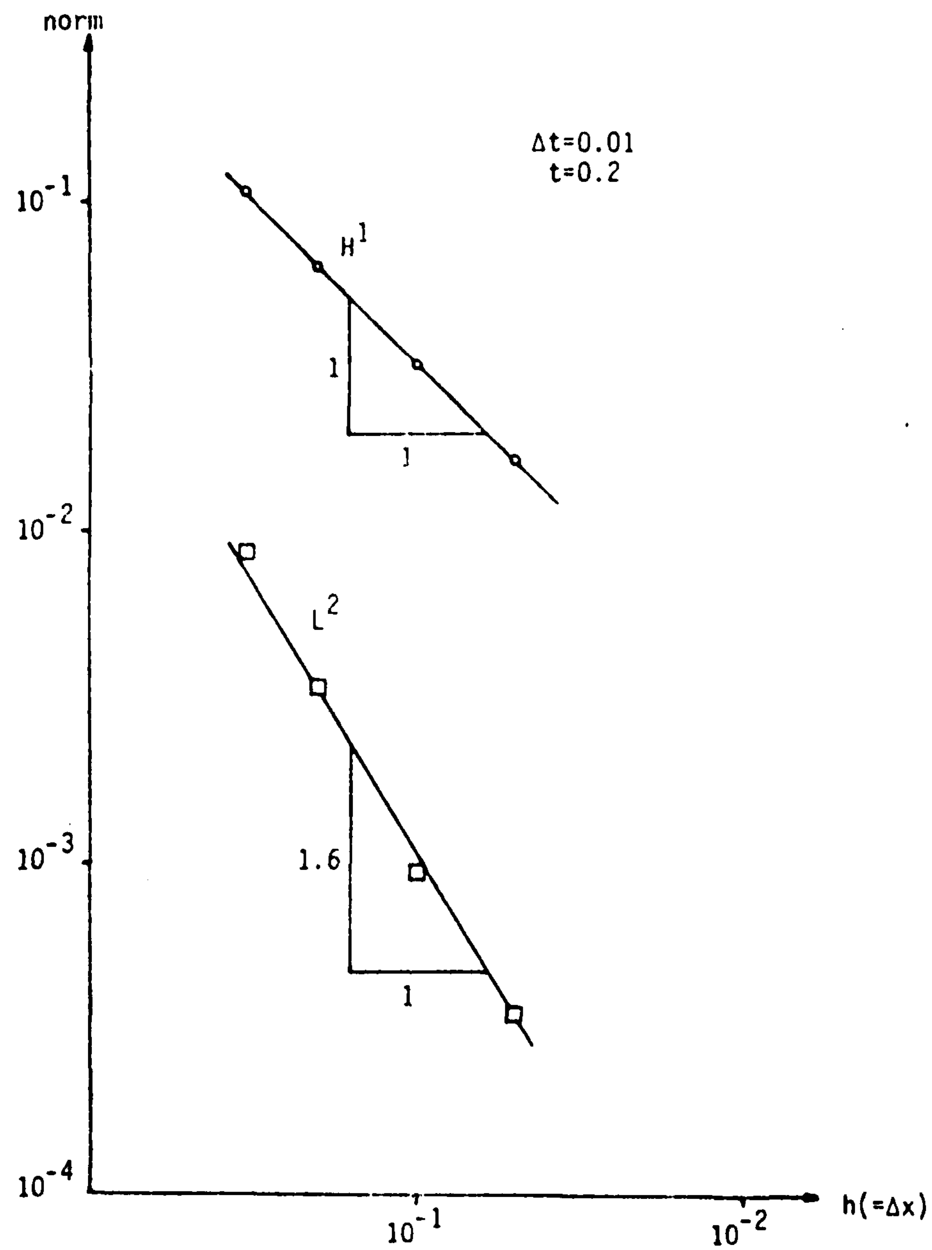


Figure 5.4. Estimate in $h(=\Delta x)$ (linear elements)

```
CONSTANTS FOR EACH FORMULATION
     PROJECTIVE SOR              MAX TOLERANCE= 1.00E-02    OMEGA= 1.600000
     LAGRANGIAN MULTIPLIER       MAX TOLERANCE= 1.00E-02    RAMDA=  .040000
                                                            OMEGA= 1.900000
     FIX POINT                   MAX TOLERANCE= 1.00E-02    ROU   =  .040000
     PENALTY METHOD              MAX TOLERANCE= 1.00E-02    EPS   = 1.00E-05
                                                            OMEGA= 1.550000
```

	PROJECT	LAGRANGE	FIX PNT	PENALTY	EXACT
1	.250000	.250000	.250000	.250000	.250000
2	.215895	.215915	.215896	.215891	.215895
3	.184290	.184330	.184289	.184282	.184289
4	.155183	.155245	.155182	.155174	.155184
5	.128576	.128662	.128574	.128566	.128579
6	.104468	.104580	.104467	.104458	.104473
7	.082880	.083000	.082860	.082851	.082888
8	.063753	.063971	.063752	.063745	.063763
9	.047146	.047344	.047145	.047138	.047157
10	.033038	.033266	.033037	.033032	.033052
11	.021430	.021695	.021430	.021425	.021447
12	.012323	.012624	.012322	.012319	.012341
13	.005715	.006055	.005715	.005713	.005736
14	.001607	.001998	.001607	.001606	.001631
15	0.000000	.000426	0.000000	-.000000	.000025
16	0.000000	.000005	0.000000	-.000000	0.000000
17	0.000000	-.000070	0.000000	-.000000	0.000000
18	0.000000	-.000186	0.000000	-.000001	0.000000
19	0.000000	-.000431	0.000000	-.000000	0.000000
20	0.000000	-.000504	0.000000	-.000000	0.000000
21	0.000000	0.000000	0.000000	-.000000	0.000000

```
PROJECTION      ITERATIONS=  20     TOLERANCE= 9.711E-06
LAGRANGE        ITERATIONS= 400     TOLERANCE= 2.624E-05
FIX POINT       ITERATIONS=  21     TOLERANCE= 9.800E-06
PENALTY         ITERATIONS=  40     TOLERANCE= 8.695E-06
```

Table 5.1 Comparison of optimization methods

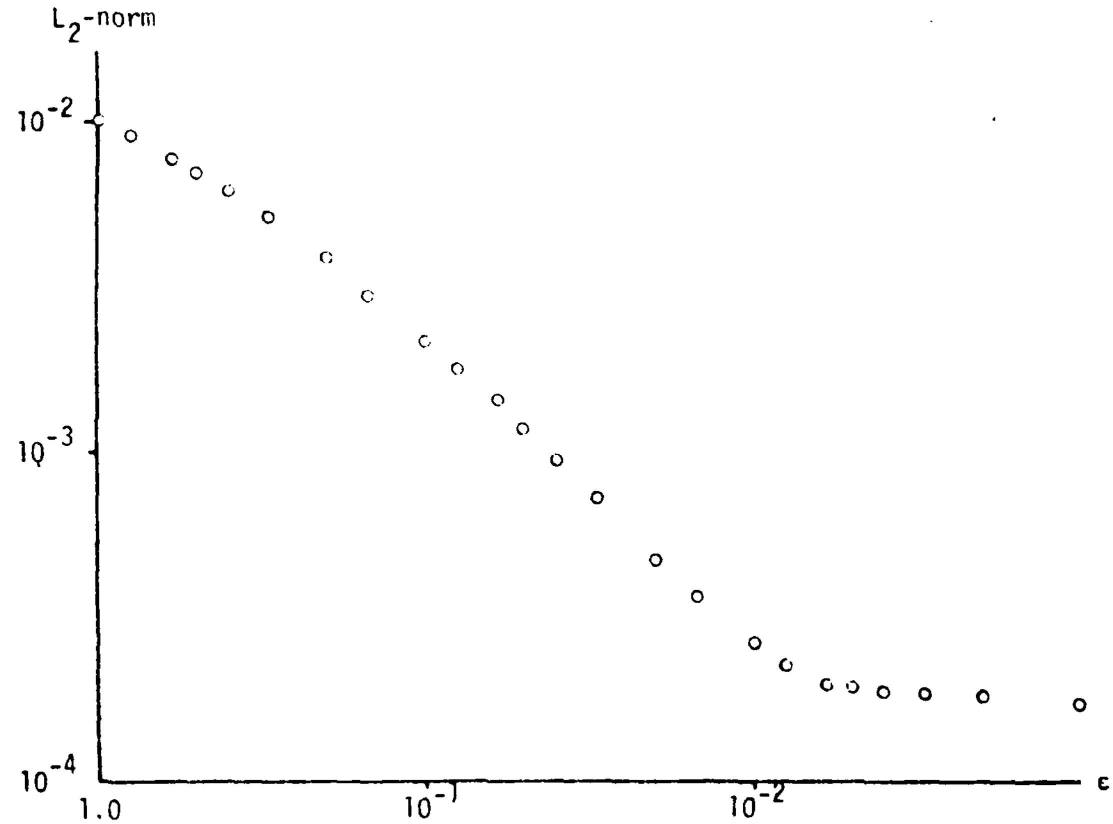


Figure 5.5 Convergence of penalty method

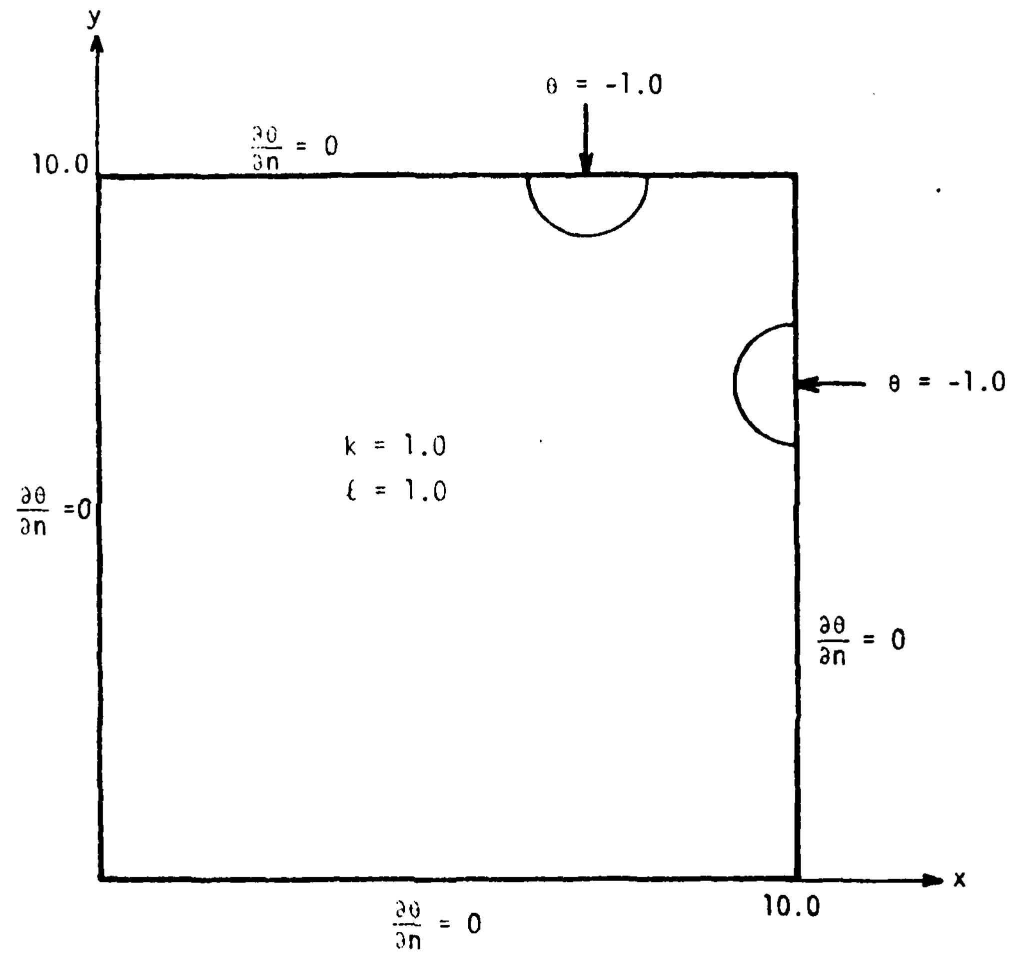


Figure 5.6 (a) Two dimensional model

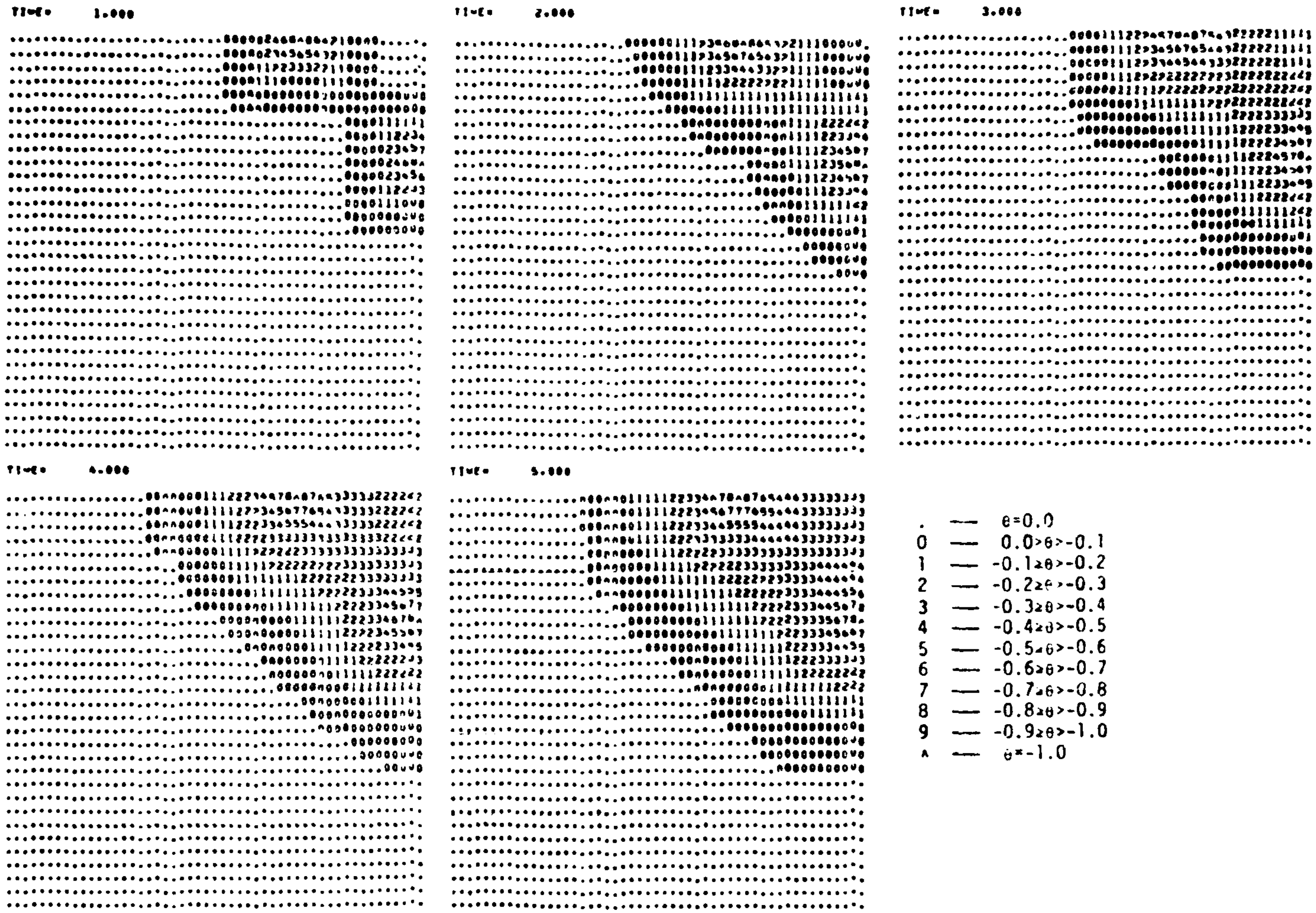


Figure 5.6 (b) Numerical results of two dimensional model

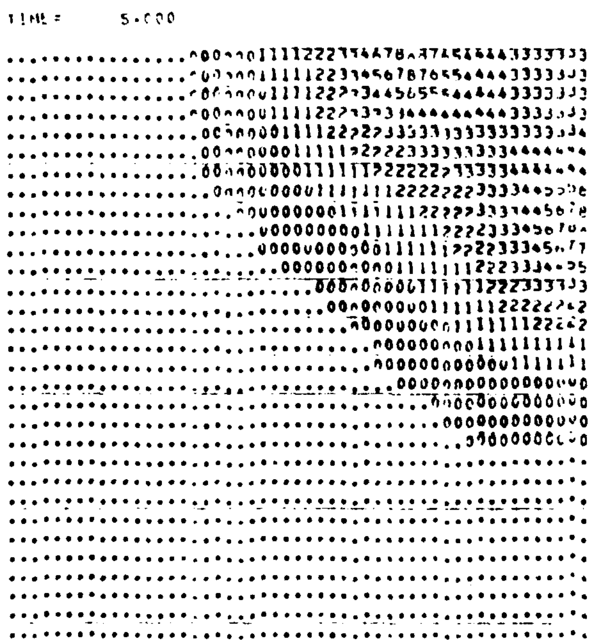


Figure 5.7 Results for $\Delta t = 0.1$

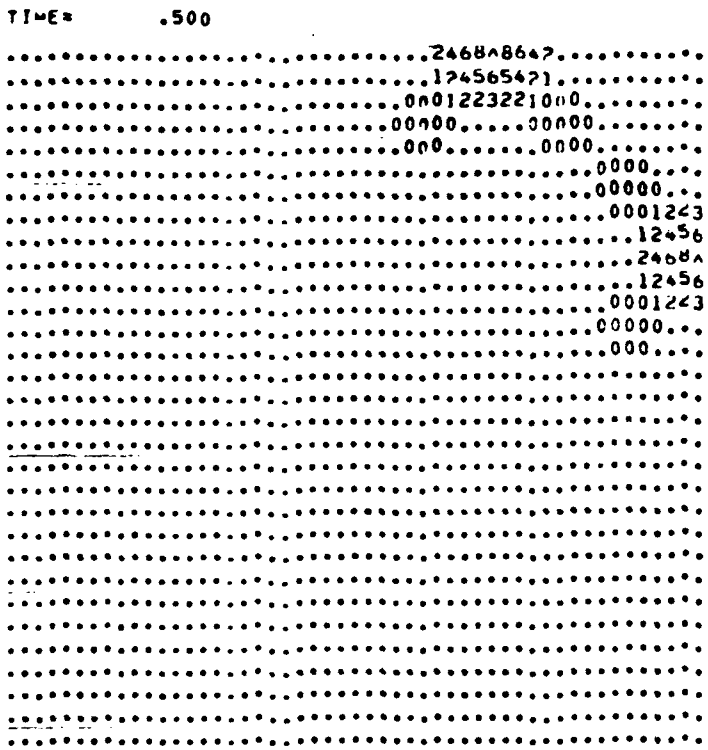
time \ ω	1.0	1.4	1.8
0.5	3	11	42
1.0	5	13	49
1.5	7	14	49
2.0	8	14	51
2.5	8	14	52
3.0	8	14	52
3.5	8	15	52
4.0	9	15	53
4.5	9	15	53
5.0	9	15	53

Table 5.2 Number of iterations for various ω of S.O.R.
($\theta = 1.0$)

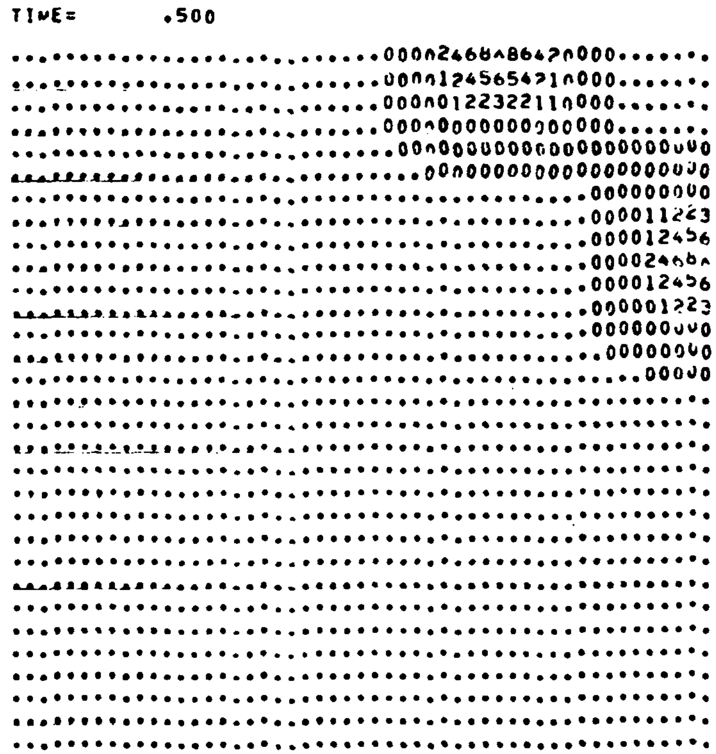
TIME= 3.5000

0	0	0	0	0	-11	-50	-100	-57	-7	-24
0	0	0	0	0	-15	5	-54	3	-54	-7
0	0	0	0	0	-8	-19	-7	-45	3	-57
0	0	0	0	0	0	0	-9	-7	-54	-100
0	0	0	0	0	0	0	0	-19	5	-50
0	0	0	0	0	0	0	0	-8	-15	-11
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0

Figure 5.8 Numerical instability for Crank-Nicolson $\theta = 0.25$

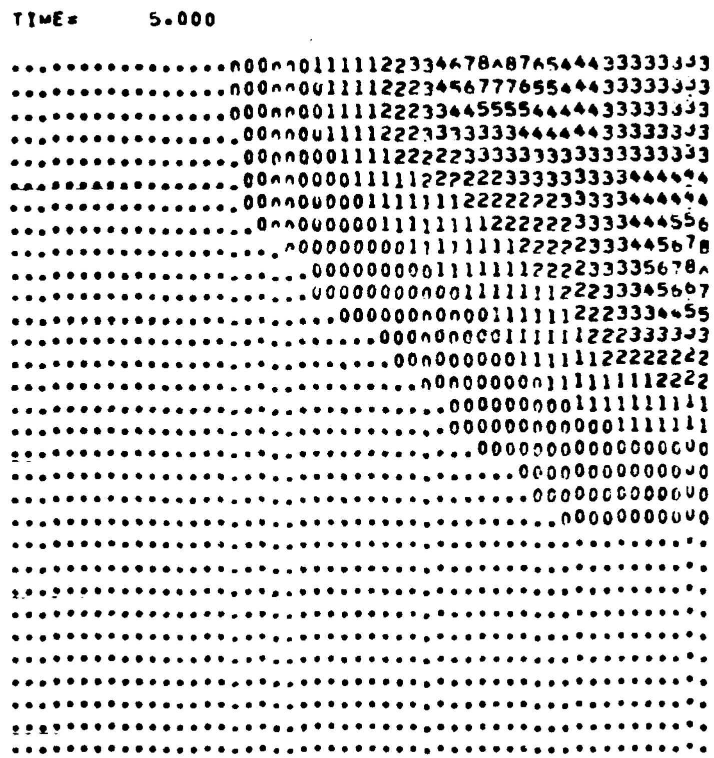


consistent mass matrix

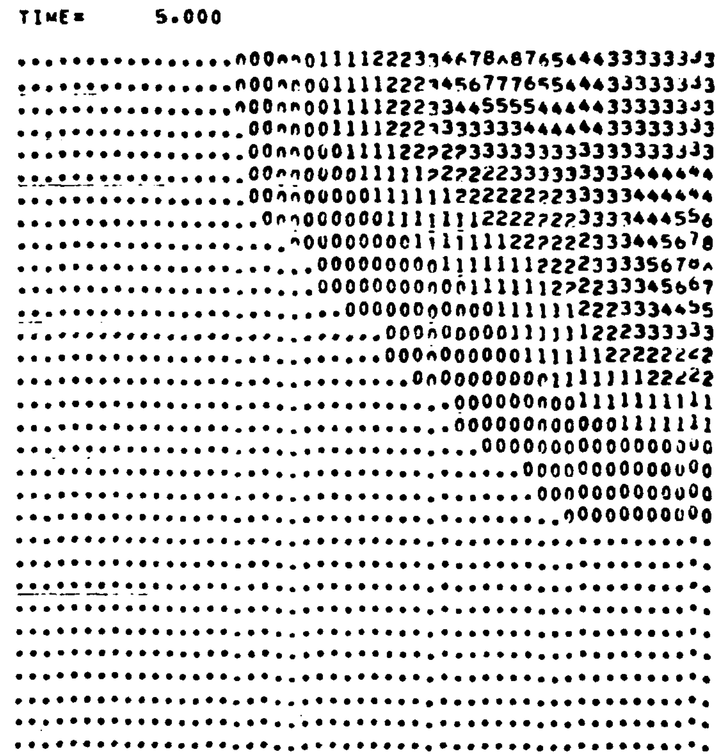


lamped mass matrix

Figure 5.9 (a) Comparison of consistent and lamped mass scheme at t = 0.5



consistent mass matrix



lamped mass matrix

Figure 5.9 (b) Comparison of consistent and lamped mass matrix at t = 5.0

APPENDIX B

Numerical Methods for Two-Phase Stefan Problems by Variational Inequalities

Appendix B

Numerical Methods for Two-Phase Stefan Problems by Variational Inequalities

1. Introduction

The problem of freezing and thawing of two- and three-dimensional ice fields under time-dependent boundary conditions can be modelled as a two-phase Stefan problem, with the free boundary, representing the interface of ice and water, unknown a priori. Relatively few effective numerical methods have been proposed for such problems, and those which attempt to treat the free boundary by schemes employing a fixed mesh are seldom given a complete mathematical justification.

The present scheme is based on the theory of Stefan problems using the freezing index which is obtained by the special transformation of the temperature field. This theory is first introduced by Duvaut [2], and studied by Frémond [3]. Its mathematical basis has been studied by Lions [5] and Aguirre-Puente and Frémond [1].

The purpose of this article is to introduce a numerical scheme for solving two-phase Stefan problems using special freezing index formulation and to discuss its efficiency. While there are several mathematical results available on the freezing index formulation, such as theorems on the existence, uniqueness, and regularity of solutions, the attempts to solve it have been limited to one-dimensional; see Aguirre-Puente and Frémond [1]. We give here multidimensional results together with some new numerical schemes.

In the following section, the field equations in terms of the freezing index are derived from the governing equations of the temperature using a special transformation. Then, a nonlinear nondifferentiable algebraic system of equations are obtained by discretizing the associated variational form of the freezing index, without specifically defining the spaces to which admissible functions belong and without discussing properties of finite dimensional subspaces used in approximations. The nonlinear system of algebraic equations is solved by a modified S.O.R. method, since it is nondifferentiable. If the system is differentiable, the Newton-Raphson method or the incremental method may be applicable, but in our case the methods are not applicable. Moreover, the nature of the nonlinearity of the system implies that some restrictions on mesh size, time increments, and physical constants such as the conductivities of the ice and water are needed, whereas the S.O.R. method may converge without any such restrictions for linear systems. We also discuss some smoothing techniques to obtain a smooth interface of ice and water and give some numerical examples. Our numerical experiments indicate that the freezing index formulation can lead to a powerful, efficient, and simple method for solving two-phase Stefan problems.

2. Two-phase Stefan Problems

2.1 A Mathematical Description. We give a brief description of a formulation of a class of two-phase Stefan problems. More detailed discussions about formulations of general two- (or one-) phase Stefan problems can be found in the monograph written by Rubinstein [6].

The case in which only the solid or the melted phase is governed by the heat equation and the temperature of the other phase remains constant, is called the one-phase Stefan problem. The two-phase Stefan problem is characterized by heat equations in both phases.

Let D be an open connected subset of R^n, $n = 1, 2, 3$, and let D be divided into two parts: the solid part D_1 and the melted part D_2. If the temperature field of the domain at time t is represented by $\theta(x,t)$, then D_1 and D_2 are defined by

$$(2.1)\quad \begin{cases} D_1(t) = \{x \in D: \theta(x,t) < 0\} \\ D_2(t) = \{x \in D: \theta(x,t) > 0\} \end{cases}$$

The surface (or maybe subregion of D) Γ_0 defined by

$$(2.2)\quad \Gamma_0(t) = \{x \in D: \theta(x,t) = 0\}$$

is called the interface (or frozen front) of the solid phase D_1 and the melted phase D_2. Let the boundary Γ of D be separated into three parts Γ_1, Γ_2, and Γ_3. The temperature field $\theta(x,t)$ is prescribed on the boundary Γ_1. There is no heat flux from the boundary Γ_2. The heat flux on the boundary Γ_3 is assumed to be proportional to the temperature on Γ_3. Then, the problem can be represented by

$$(2.3)\quad \begin{cases} C_1\dot{\theta} = \nabla\cdot(k_1\nabla\theta) & \text{in } D_1 \\ C_2\dot{\theta} = \nabla\cdot(k_2\nabla\theta) & \text{in } D_2 \end{cases}$$

$$(2.4)\quad \begin{cases} \theta(x,t) = g(x,t) & \text{on } \Gamma_1 \\ \partial_i\theta(x,t) = 0 & \text{on } \Gamma_2 \cap \bar{D}_i \\ \partial_i\theta(x,t) = -p_i\theta(x,t) + q_i(x,t) & \text{on } \Gamma_3 \cap \bar{D}_i \end{cases}$$

$$(2.5)\quad \begin{cases} \theta(x,t) = 0 & \text{on } \Gamma_0 \\ [k\nabla\theta]\cdot\nabla S + \ell = 0 & \text{on } \Gamma_0 \end{cases}$$

$$(2.6)\quad \theta(x,0) = \theta_0(x) \qquad \text{on } D$$

Here C_i and k_i, $i = 1,2$, are the mass heat capacity and the heat conductivity of i-th phase, respectively, g, p_i, and q_i are given proper functions, θ_0 is the initial temperature of the domain, $\dot{\theta} = \partial\theta/\partial t$, $\partial_i\theta = k_i \sum_{\alpha=1}^{n} n_\alpha(\partial\theta/\partial x_\alpha)$, $i = 1,2$, $n = (n_1,\ldots,n_n)$ is the outward normal unit vector on Γ, and

$$[\![k\nabla\theta]\!] = k_2(\nabla\theta)^+ - k_1(\nabla\theta)^- \tag{2.7}$$

where $(\psi)^+$ is the limit of ψ on Γ_0 coming from D_2, and $(\psi)^-$ is the limit of ψ on Γ_0 coming from D_1. The function $t = S(x)$ indicates the position of Γ_0, which is sometimes written by $x = L(t)$. The value ℓ is the latent heat of the solid phase.

We remark that the portion of Γ_0 is unknown a priori, and that the gradient $\nabla\theta$ of the temperature field is discontinuous on Γ_0.

2.2 The Freezing Index

Because of the discontinuity of the gradient of the temperature on the interface Γ_0, the problem cannot be formulated variationally in the whole domain D for the temperature field if the position of the interface Γ_0 is unknown. To avoid this difficulty, Duvaut [2] introduces a special transformation, which is later called the freezing index by Frémond [3]:

$$u(x,t) = \int_0^t k_i\theta(x,\tau)\, d\tau \tag{2.8}$$

where

$$i = 1 \text{ if } x \in D_1(\tau), \quad i = 2 \text{ if } x \in D_2(\tau) \tag{2.9}$$

Since k_i, $i = 1,2$ are constants, $u(x,\cdot)$ is differentiable if $\theta(x,\cdot)$ is continuous. More generally, if $\theta(x,\cdot)$ is measurable, $u(x,\cdot)$ is differentiable in generalized sense. Then

$$\theta(x,t) = \frac{1}{k_i}\dot{u}(x,t) \tag{2.10}$$

This implies that

$$(2.11)\qquad \begin{cases} D_1(t) = \{x \in D:\ \dot{u}(x,t) < 0\} \\ D_2(t) = \{x \in D:\ \dot{u}(x,t) > 0\} \end{cases}$$

since $k_i > 0$. Under the assumption that k_1 and k_2 are constants, (2.3) implies

$$\nabla\cdot\nabla u(x,t) = \begin{cases} -C_1\theta(x,0) + C_1(x,t) & \text{if } x \in D_1(0) \cap D_1(t) \\ -C_1\theta(x,0) + C_2(x,t) + \ell & \text{if } x \in D_1(0) \cap D_2(t) \\ -C_2\theta(x,0) + C_2(x,t) & \text{if } x \in D_2(0) \cap D_2(t) \\ -C_2\theta(x,0) + C_1(x,t) - \ell & \text{if } x \in D_2(0) \cap D_1(t) \end{cases}$$

Applying (2.6) and (2.10),

$$(2.12)\qquad \frac{C_i}{k_i}\,\dot{u}(x,t) - \nabla\cdot\nabla u(x,t) = C_j\theta_0(x) + \epsilon_{ij}\ell$$

where

$$(2.13)\qquad \begin{cases} i = 1 \text{ if } x \in D_1(t), & i = 2 \text{ if } x \in D_2(t) \\ j = 1 \text{ if } x \in D_1(0), & j = 2 \text{ if } x \in D_2(0) \end{cases}$$

$$(2.14)\qquad \epsilon_{12} = 1, \quad \epsilon_{21} = -1, \quad \epsilon_{11} = \epsilon_{22} = 0$$

From (2.8),

$$\partial u(x,t) = \int_0^t k_i \nabla\theta(x,\tau)\cdot n(x)\, d\tau = \int_0^t \partial_i\theta(x,\tau)\, d\tau$$

Then, putting

$$(2.15)\qquad \hat{g}(x,t) = \int_0^t k_i g(x,t)\, d\tau, \quad i = 1 \text{ if } g(x,\tau) < 0, \quad i = 2 \text{ if } g(x,\tau) > 0$$

$$(2.16)\qquad \hat{q}(x,t) = \int_0^t q_i(x,\tau)\, d\tau, \ i = 1 \text{ if } x \in D_1(\tau), \ i = 2 \text{ if } x \in D_2(\tau),$$

boundary conditions (2.4) can be transformed to

$$(2.17) \qquad u(x,t) = \hat{g}(x,t) \quad \text{on } \Gamma_1$$

$$(2.18) \qquad \partial u(x,t) = 0 \quad \text{on } \Gamma_2$$

and

$$(2.19) \qquad \partial u(x,t) = -\frac{p_i}{k_i} u(x,t) + \hat{q}(x,t) \quad \text{on } \Gamma_3$$

where

$$i = 1 \text{ if } x \in D_1(t), \quad i = 2 \text{ if } x \in D_2(t)$$

Here we have already counted the initial condition of u, i.e.,

$$(2.20) \qquad u(x,0) = 0 \quad \text{in } D$$

The interface conditions (2.5) have been also taken into account in the above considerations. Therefore, the two-phase Stefan problem (2.3)-(2.6) is transformed to the field equation (2.12), the boundary conditions (2.17), (2.18), and (2.19), and the initial condition (2.20) in terms of the freezing index.

3. Discrete Two-phase Stefan Problems

We will now consider discrete problems associated with the problem {(2.12), (2.17), (2.18), (2.19), (2.20)} using finite difference and element methods. For details of mathematical analysis of the same class of two-phase Stefan problems, see Lions [3], and Aguirre-Puente and Frémond [1].

3.1 A Variational Formulation

Under the presumption that $\dot{u}(x,t)$ belong to $L^2(D)$ at every $t \in [0,T]$, $t \in \mathbb{R}$,

$$\int_D (d_i \dot{u} v + \nabla u \cdot \nabla v)\, dx$$

$$= \int_{\Gamma_2} (\partial u) v\, ds + \int_{\Gamma_3} (\partial u) v\, ds + \int_D (d_i \dot{u} - \nabla \cdot \nabla u) v\, dx$$

for every v such that $v = \hat{g}$ on Γ_1, here $d_i = C_i/k_i$. By (2.12), (2.18), and (2.19),

$$\int_D (d_i \dot{u} v + \nabla u \cdot \nabla v)\, dx = \int_{\Gamma_3} (-e_i u + \hat{q}) v\, ds + \int_D (C_j \theta_0 v + \epsilon_{ij} \ell v)\, dx$$

where $e_i = p_i/i_k$. That is, putting

$$(u,v) = \int_D uv\, dx$$

(3.1) $$a_i(u,v) = \int_D \nabla u \cdot \nabla v\, dx + \int_{\Gamma_3} e_i uv\, ds$$

$$L_j(v) = \int_D C_j \theta_0 v\, dx + \int_{\Gamma_3} \hat{q} v\, ds$$

we have, for every $t \in (0,T]$,

(3.2) $$u \in K(t):\ (d_i \dot{u}, v) + a_i(u,v) = (\epsilon_{ij} \ell, v) + L_j(v)$$

for every $v \in K_0(t)$ with the initial condition

(3.3) $$u(x,0) = 0$$

where

(3.4) $$K(t) = \{v(t) \in H^1(D):\ v(t) = \hat{g}(t) \text{ a.e. on } \Gamma_1\}$$

(3.5) $$K_0(t) = \{v(t) \in H^1(D):\ v(t) = 0 \text{ a.e. on } \Gamma_1\}$$

3.2 Finite Difference Methods

Suppose that the domain $D \subset \mathbb{R}^2$ is a rectangle which is covered by the uniform net Σ_D. Let Σ be the set of all nodal points interior of the domain. Let Σ_1, Σ_2, and Σ_3 be sets of all nodal points on Γ_1, Γ_2, and Γ_3, respectively. For simplicity, Σ_2 and Σ_3 are assumed to be null. Let the particular nodal point of the net be represented by the pair (α,β), which indicates the position of the nodal point, i.e., the coordinates of the point are given by $(\alpha\Delta x, \beta\Delta y)$, where Δx and Δy are intervals of nodal points in x and y directions, respectively. In this article, for simplicity, $\Delta x = \Delta y = \Delta$. Then, the variational problem (3.2) is reduced to the nonlinear system:

$$d_i^{\alpha,\beta}u_{\alpha,\beta} + \frac{4}{\Delta^2}u_{\alpha,\beta} = \frac{1}{\Delta^2}(u_{\alpha-1,\beta} + u_{\alpha+1,\beta} + u_{\alpha,\beta-1} + u_{\alpha,\beta+1}) + c_j^{\alpha,\beta}(\theta_0)_{\alpha,\beta} + \epsilon_{ij}^{\alpha,\beta}\ell \tag{3.6}$$

for $(\alpha,\beta) \in \Sigma$, and

$$u_{\alpha,\beta} = \hat{g}_{\alpha,\beta} \tag{3.7}$$

for $(\alpha,\beta) \in \Sigma_1$. Here the summation convention is not applied.

Nonlinearities can be included in $d_i^{\alpha,\beta}$ and $\epsilon_{ij}^{\alpha,\beta}$, but these make it necessary to resolve the form (3.6).

Now, the approximation of $d_i^{\alpha,\beta}\,\dot{u}_{\alpha,\beta}$ in time derivative is given by

$$d_i^{\alpha,\beta}\dot{u}_{\alpha,\beta}(n\Delta t) = d_i^{\alpha,\beta}(n\Delta t)\,\frac{1}{\Delta t}\,(u_{\alpha,\beta}^{n} - u_{\alpha,\beta}^{n-1}) \tag{3.8}$$

where $u_{\alpha,\beta}^{n} = u_{\alpha,\beta}(n\Delta t)$, and Δt is the given time interval. Then (3.6) can be solved by a kind of S.O.R. method:

$$\text{(3.9)}\quad \begin{cases} \text{(i)}\quad \text{put}\quad u_{\alpha,\beta}^{n}(0) = u_{\alpha,\beta}^{n-1} \\[2ex] \text{(ii)}\quad R_{\alpha,\beta}^{n}(m) = \frac{1}{4}\,(u_{\alpha-1,\beta}^{n}(m) + u_{\alpha+1,\beta}^{n}(m-1) + u_{\alpha,\beta-1}^{n}(m) \\ \qquad\qquad + u_{\alpha,\beta+1}^{n}(m-1) + \Delta^2 c_j^{\alpha,\beta}(\theta_0)_{\alpha,\beta}) \\[2ex] \qquad N_{\alpha,\beta}^{n}(m) = \epsilon_{ij}^{\alpha,\beta}(n,m)\,\ell - d_i^{\alpha,\beta}(n,m)\frac{1}{\Delta t}\,(u_{\alpha,\beta}^{n}(m-1) - u_{\alpha,\beta}^{n-1}) \\[1ex] \qquad u_{\alpha,\beta}^{n}(m) = (1-\omega)\,u_{\alpha,\beta}^{n}(m-1) + \omega(R_{\alpha,\beta}^{n}(m) + \frac{\Delta^2}{4}\,N_{\alpha,\beta}^{n}(m)) \\[2ex] \text{(iii)}\quad \text{Repeat until} \\[1ex] \qquad\qquad \sum|u_{\alpha,\beta}^{n}(m) - u_{\alpha,\beta}^{n}(m-1)|\,/\,\sum|u_{\alpha,\beta}^{n}(m)| < \epsilon \end{cases}$$

where $u_{\alpha,\beta}^{n}(m)$ means the value of $u_{\alpha,\beta}^{n}$ at m-th iteration of the S.O.R. method, ω is the iteration factor which is expected to be in the interval $(0,2)$, ϵ is the admissible error of the convergence of iterations, and indices i and j of $\epsilon_{i,j}^{\alpha,\beta}(n,m)$ and $d_i^{\alpha,\beta}(n,m)$ are

$$\begin{cases} i = 1 \text{ if } \dot{u}_{\alpha,\beta}^{n}(m) < 0 \\ i = 2 \text{ if } \dot{u}_{\alpha,\beta}^{n}(m) > 0 \end{cases} \quad \begin{cases} j = 1 \text{ if } (\theta_0)_{\alpha,\beta} < 0 \\ j = 2 \text{ if } (\theta_0)_{\alpha,\beta} > 0 \end{cases} \tag{3.10}$$

where

$$\dot{u}^n_{\alpha,\beta}(m) = \frac{1}{\Delta t}\,(u^n_{\alpha,\beta}(m-1) - u^{n-1}_{\alpha,\beta})$$

The nonlinear term $\epsilon^{\alpha,\beta}_{ij}(n,m)\,\ell$ in (3.9) is varies like a step function, while other remaining terms in (3.9) are expected to change moderately. This implies that the relationship

$$(3.11) \qquad \left|R^n_{\alpha,\beta}(m) - \frac{\Delta^2}{4}\, d^{\alpha,\beta}_i(n,m)\,\dot{u}^n_{\alpha,\beta}(m)\right| \gg \left|\frac{\Delta^2}{4}\,\epsilon^{\alpha,\beta}_{i,j}(n,m)\,\ell\right|$$

must be satisfied in order to get convergence of the iterative scheme (3.9).

EXAMPLE 1. Let us consider the one-dimensional problem, whose domain D is the interval (0,1), material constants d_1 and d_2 (i.e., k_1, k_2, C_1, and C_2) are given constants. Let the initial temperature θ_0 be given by

$$\theta_0(x) = (x - 1/2) \quad \text{if} \quad x \le 1/2, \quad = 0 \quad \text{if} \quad x > 1/2$$

The boundary conditions $g(0,t)$ and $g(1,t)$ are given by

$$g(0,t) = -1/2 \qquad g(1,t) = t$$

Then, for $t < 1$,

$$\hat{g}(0,t) = -k_1 t/2\,, \qquad \hat{g}(1,t) = k_2 t^2/2$$

If $n = 1$, and $\alpha\lambda = 0.5$, (3.11) becomes

$$\left|0.5(u^1_{\alpha-1}(m) + u^1_{\alpha+1}(m-1)) - 0.25\Delta^2 d^{\alpha}_1(1,m)\,\dot{u}^1_{\alpha}(m)\right| \gg 0.25\Delta^2\ell$$

Under the assumption that $\dot{u}^1_\alpha$ is almost the same with $\dot{u}^0_\alpha = \theta_{0\alpha}$ (i.e., the time interval Δt is taken to be sufficiently small), this becomes

$$0.5k_1\Delta \cdot \Delta t \gg 0.25\Delta^2 \ell \quad \text{i.e.,}$$

(3.12) $$2k_1 \Delta t \gg \Delta \ell$$

That is, in order to satisfy the condition (3.11), the relationships (3.12) has to be assumed. Under this condition, the iterative scheme (3.9) may converge.

For the case that $C_1 = 0.5$, $C_2 = k_1 = k_2 = 1.0$, $\ell = 100$, and $\Delta = 0.02$, the convergence for the various time increments Δt is obtained in Table 1.

In Figure 1, the position of the interface is described for several time increments Δt. This shows that the time increment Δt has to be small enough in order to treat the position of the interface. That is, it is preferable to use a Δt which is the almost-limit value for convergence of the iterative scheme (3.9).

<u>EXAMPLE 2</u>. Let us again consider a one-dimensional, two-phase Stefan problem whose material constants C_1, C_2, k_1, k_2, and ℓ are obtained for a silty soil with twenty-percent moisture content, i.e.

$k_1 = 60$ kcal/m·day.°C $k_2 = 50$ kcal/m·day.°C

$C_1 = 450$ kcal/m^3·°C $C_2 = 600$ kcal/m^3·°C

$\ell = 24000$ kcal/m^3

Initially, the soil foundation is unfrozen so that the initial temperature θ_0 is given by

$$\theta_0 = ax^2, \qquad a = g(L,0)/L^2$$

where x is the depth of the foundation, and $g(L,0)$ is the initial temperature at the end depth L. We specify boundary conditions $g(0,t)$ and $g(L,t)$ described in Table 2. Let the depth of foundation be given by $L = 5m$. We use a 3-point finite difference scheme for space discretization with the mesh length Δ. For the time increment, $\Delta t = 10$ days is used.

By arguments similar to those in Example 1, the criteria (3.11) becomes

$$\frac{1}{4}|k_1 \frac{\Delta^2}{2} g(0,\Delta t)| >> \frac{1}{4} \Delta^2 \ell \tag{3.13}$$

Since the frozen front propagates from the surface of the foundation $x = 0$, the condition (3.11) has to be evaluated at $\alpha = 1$ and $n = 1$. Since $(\theta_0)_1$ is almost zero at $x = \alpha\Delta$, the relation (3.13) is obtained under the assumption that u_1^1 is small enough so that $\Delta^2 d_i \dot{u}_1^1$ is negligible. Here $g(0,t)$ is the boundary temperature given on the surface of the foundation $x = 0$.

We calculate three cases, i.e., $\Delta = 0.1$, $\Delta = 0.2$, and $\Delta = 0.5$. As shown in Table 3, convergence of the scheme (3.9) is not obtained for the case of $\Delta = 0.5$. For $\Delta = 0.2$, the scheme (3.9) is almost convergent. That is, around the interface, values of the freezing index vary periodically, and the criterion $(3.9)_{iii}$ is not satisfied, choosing $\epsilon = 10^{-3}$. However, the case of $\Delta = 0.1$ gives nice stable convergence of (3.9) except for the first few time steps. For the first few steps, the relative tolerance may not reach the given criterion $\epsilon = 10^{-3}$, since the value of the freezing index is considerably smaller than the latent heat ℓ. Recall that by a phase change, the latent heat ℓ enters the force term in (3.9). However, after several steps, the global result becomes stable and the large relative tolerance for the first few steps does not affect the subsequent results. If we do not expect large relative tolerances, they can be avoided by shifting the value of the initial freezing index. That is, in place that the freezing index is assumed to be zero at the initial stage, we just shift $u(x,0)$ to some positive value u_0 whose order of magnitude may be the same as the latent heat ℓ.

We also remark that the condition (3.13) indicates that the mesh size has to be so small that the frozen front can exceed at least one mesh if the system is far from equilibrium.

We shall discuss some modifications of the approximation (3.6) in order to get an efficient method of solving (3.6).

First, the term $d_i^{\alpha,\beta}\dot{u}_{\alpha,\beta}(n\Delta t)$ is approximated by

$$(3.14)\qquad d_i^{\alpha,\beta}\dot{u}_{\alpha,\beta}(n\Delta t) = d_i^{\alpha,\beta}((n-1)\Delta t)\,\frac{1}{\Delta t}\,(u^n_{\alpha,\beta} - u^{n-1}_{\alpha,\beta})$$

instead of (3.8). Then, the coefficient $d_i^{\alpha,\beta}$ can be fixed at each time step. That is, material constants at n-th step are determined by values at $(n-1)$-th step which have been already obtained. This implies the modification of (3.9)-(ii):

$$(3.9)'\quad \begin{cases} \text{(ii)}'\quad R^n_{\alpha,\beta}(m) = \left(\dfrac{\Delta^2}{\Delta t}\, d_i^{\alpha,\beta}((n-1)\Delta t) + 4\right)^{-1}\Big(u^n_{\alpha-1,\beta}(m) + u^n_{\alpha+1,\beta}(m-1) \\ \qquad\qquad + u^n_{\alpha,\beta-1}(m) + u^n_{\alpha,\beta+1}(m-1) + \Delta^2 C_j^{\alpha,\beta}\theta_{0\alpha,\beta} \\ \qquad\qquad + \dfrac{\Delta^2}{\Delta t}\, d_i^{\alpha,\beta}((n-1)\Delta t)\, u^{n-1}_{\alpha,\beta}\Big) \\ N^n_{\alpha,\beta}(m) = \epsilon^{\alpha,\beta}_{ij}(n,m)\,\ell \\ u^n_{\alpha,\beta}(m) = (1-\omega)\,u^n_{\alpha,\beta}(m-1) + \omega\left(R^n_{\alpha,\beta}(m) + \dfrac{\Delta^2}{4} N^{\alpha}_{\alpha,\beta}(m)\right) \end{cases}$$

EXAMPLE 3. Here the problem described in Example 1 is solved by the iterative procedure (3.9)' instead of (3.9). Let C_1 0.5, C_2 $k_1 = k_2 = \ell = 1.0$, and let the boundary conditions be given by

$$g(0,t) = (t-1)/2, \qquad g(1,t) = t$$

The initial temperature θ_0 is same as Example 1. Results, shown in are obtained by the mesh size $\Delta = 0.02$, the time increment $\Delta t = 0.05$, the iteration factor $\omega = 1.0$ in (3.9), and the tolerance $\epsilon = 10^{-3}$.

According to numerical calculations, see Figure 3, (3.9) and the modified scheme (3.9)' give almost the same propagation of the solid phase. However, after reaching the limit of propagation of the solid phase, the modified scheme (3.9)' gives considerably different results from (3.9). This difference comes from $O(d_1) = O(\ell)$, i.e., the order of the latent heat ℓ is almost the same as the one of $d_1 = C_1/k_1$ and $d_2 = C_2/k_2$. If ℓ is much bigger than d_i,, then differences of results by (3.9) amd (3.9)' may not be so large.

We also solved the same problem by the method given by Nogi [7]. Details of this comparison are found in Kikuchi [4].

Second, the term $\epsilon_{ij}^{\alpha,\beta}\ell$ of (3.6), which is related with the latent heat of the solid phase, is homogenized by

$$\text{(3.15)} \qquad {}_H\epsilon_{ij}^{\alpha,\beta}\ell = \{h_1(\epsilon_{ij}^{\alpha-1,\beta} + \epsilon_{ij}^{\alpha+1,\beta} + \epsilon_{ij}^{\alpha,\beta-1} + \epsilon_{ij}^{\alpha,\beta+1}) + h_2\epsilon_{ij}^{\alpha,\beta}\}\ell/(4h_1+h_2)$$

for proper number h_1 and h_2. Then, in (3.9)-(ii), the term $\epsilon_{ij}^{\alpha,\beta}(n,m)\ell$ is replaced by ${}_H\epsilon_{ij}^{\alpha,\beta}\ell$.

EXAMPLE 4. A two-dimensional model is considered in this example. Let $D = (0,0.4) \times (0,0.4)$, $C_1 = 0.5$, $C_2 = k_1 = k_2 = \ell = 1.0$. The initial temperature θ_0 is given by $\theta_0 = 0$ in D. Boundary conditions are given by

$$
\begin{aligned}
g(0,y) &= (0.8 - y)(2t - 1) \\
g(x,0.4) &= (0.8 - x)(2t - 1) \\
g(0.4,y) &= 0.5\sqrt{y}\,(1-3t) \\
g(x,0) &= 0.5\sqrt{x}\,(1-3t)
\end{aligned}
$$

The uniform mesh is employed for the net of finite difference with $\Delta = \Delta x = \Delta y = 0.02$. The time increment Δt is 0.05. These satisfy the criteria (3.12) and (3.13), i.e.,

$$
\begin{aligned}
2k_1 \Delta t &= 0.1 \gg \Delta\ell = 0.02 \\
2k_1 \Delta t &= 0.1 \gg \Delta^2 \ell = 0.0004
\end{aligned}
$$

At the 9-th and 9-th time step, i.e., at $t = 0.4$ and $t = 0.45$, phase transition from the melted region to the solid region becomes very sensitive in this example. At the 7-th time step, the range $\{(x,y): 0 < x < 0.4,\ 0 < y < 0.4,$ and $y > x\}$ is melted with almost zero degree temperature. Then, during the 7-th step to 8-th step, the boundaries $(x,0)$ and $(0.4,y)$ become solid. Then, the solid phase develops gradually from the boundaries and there remains the melted region inside the model. We examine how the isolated melted region remains at $t = 0.4$ and $t = 0.45$ using the modification (3.15) for various factors h_1 and h_2. Four cases are shown in Figure 4-(a) (at $t = 0.4$) and Figure 4-(b) (at $t=0.45$).

Except for line of zero temperature, equi-contour lines coincide for any choice of h_1 and h_2 at $t = 0.4$ and 0.45. That is, the homogenization (3.15) does not affect the temperature field except around the frozen front.

However, the position of the phase transition depends strongly upon the choice of h_1 and h_2. Numerical results show that equi-distributed homogenization ($h_1 = h_2 = 1$) gives a fairly smooth interface of the solid and melted phases.

Third, the conditions (3.10) may be replaced by

$$(3.10)' \qquad \begin{cases} i = 1 & \text{if} \quad \dot{u}^n_{\alpha,\beta}(m) < -\epsilon_1 k_1 \\ i = 2 & \text{if} \quad \dot{u}^n_{\alpha,\beta}(m) > \epsilon_2 k_2 \end{cases}$$

for the term $\epsilon^{\alpha,\beta}_{ij}(n,m)$ in (3.9), where ϵ_1 and ϵ_2 are given small positive numbers. The domain

$$(3.16) \qquad T_R = \{x \in D: -\epsilon_1 k_1 \leq \dot{u}(x,t) \leq \epsilon_2 k_2\}$$

might be called the transient region of the solid and melted phases.

<u>EXAMPLE 5</u>. The same example described in Example 4 is solved by applying (3.10)' instead of (3.10) in (3.9). Here $h_1 = 0$ and $h_2 = 1$ are taken, i.e., no homogenizations are made. As Example 4, numerical results at $t = 0.4$ are shown in Figure 5. In the case of $\epsilon_1 = \epsilon_2 = 10^{-2}$, shown in 5-(c), the whole domain becomes solid. However, the transient region, indicated by small circles, spreads widely. In this range, it cannot be precisely determined whether the point is melted or frozen. We may say that intermediate state occurs in that range. The reason the whole domain becomes solid is that the body force due to the latent heat is entirely neglected in the transient region. In the case of $\epsilon_1 = \epsilon_2 = 10^{-3}$ as shown in Figure 5-(B), the transient region certainly becomes narrower than the case of $\epsilon_1 = \epsilon_2 = 10^{-2}$.

The temperature fields away from zero degree almost coincide in both cases. This means that the effects of the modification (3.10)' are limited to regions around the phase change.

3.3 Finite Element Methods

For cases where the boundary conditions are not only the Dirichlet type but also the Neumann and the third types, or their domains are considerably irregular, finite element discretization is preferable. Let the domain D be triangulated. Let Σ be the set of all nodal points in D and on Γ_2. Let Σ_1 and Σ_3 be sets of all nodal points on Γ_1 and Γ_3, respectively. Let ϕ_α be the global interpolation function at α-th nodal point which is constructed by local interpolation functions (shape functions) attached to finite elements. Then, every function $v(t)$ in $H^1(\Omega)$ can be approximated by

$$v(x,t) = v^\alpha(t)\ \phi_\alpha(x) \tag{3.17}$$

Here the summation convention is applied. In this section, this convention is used throughout--all repeated indices are summed throughout their range. Putting $\dot{v}^\alpha = dv^\alpha/dt$, (3.2) is discretized by

$$\{u^\alpha\} \in K_h(t):\ \dot{u}^\alpha M^i_{\alpha\beta} v^\beta + u^\alpha S^i_{\alpha\beta} v^\beta = F^{ij}_\beta v^\beta + L^j_\beta v^\beta \text{ for every } \{v^\beta\} \in K^0_h(t) \tag{3.18}$$

$$K_h(t) = \{\{v^\alpha(t)\} \in \mathbb{R}^N:\ v^\alpha(t) = \hat{g}^\alpha(t) \qquad \alpha \in \Sigma_1\} \tag{3.19}$$

$$(3.20)\qquad K_h^0(t) = \{(v^\alpha(t)) \in \mathbb{R}^N:\ v^\alpha(t) = 0 \quad \alpha \in \Sigma_1\}$$

Recall that the index i depends upon the current value of $\dot{u}^\alpha$, and the index j depends upon the initial value of $\dot{u}^\alpha$, i.e., θ_0^α. More precisely,

$$(3.21)\qquad \begin{cases} i = 1 & \text{if } \dot{u}^\alpha(t) < 0 \\ i = 2 & \text{if } \dot{u}^\alpha(t) > 0 \end{cases} \qquad \begin{cases} j = 1 & \text{if } \theta_0^\alpha < 0 \\ j = 2 & \text{if } \theta_0^\alpha > 0 \end{cases}$$

Matrices $M^i_{\alpha\beta}$ and $S^i_{\alpha\beta}$ are defined by

$$(3.22)\qquad M^i_{\alpha\beta} = (d_i \phi_\alpha, \phi_\beta), \qquad S^i_{\alpha\beta} = a_i(\phi_\alpha, \phi_\beta)$$

Vectors F^{ij}_β and L^j_β are defined by

$$(3.23)\qquad F^{ij}_\beta = (\epsilon_{ij}\ell, \phi_\beta), \qquad L^j_\beta = L_j(\phi_\beta)$$

Here $(\cdot,\cdot)$, $a_i(\cdot,\cdot)$, and $L_j(\cdot)$ have been defined in (3.1). Then, from (3.18), it is necessary to solve the following nonlinear system

$$(3.24)\qquad \{u^\alpha\} \in K_h(t):\ \dot{u}^\alpha M^i_{\alpha\beta} + u^\alpha S^i_{\alpha\beta} = F^{ij}_\beta + L^j_\beta$$

This nonlinear system can be treated by the iterative algorithm described in (3.9). Similar with the case of finite difference methods, modifications on $M^i_{\alpha\beta}$, $S^i_{\alpha\beta}$, and F^{ij}_β can be considered.

First, matrices $M^i_{\alpha\beta}$ and $S^i_{\alpha\beta}$ may be replaced by

$$(3.25)\qquad \begin{cases} M^i_{\alpha\beta}(n\Delta t) = (d_i((n-1)\Delta t)\phi_\alpha, \phi_\beta) \\ S^i_{\alpha\beta}(n\Delta t) = a_i(\phi_\alpha, \phi_\beta)((n-1)\Delta t) \end{cases}$$

Then matrices $M^i_{\alpha\beta}$ and $S^i_{\alpha\beta}$ do not depend upon the current value of $\dot{u}^\alpha(n\Delta t)$, i.e., nonlinearity of matrices are disappeared at each time step.

However, the homogenization discussed in (3.15) is difficult for the finite element discretization. So we consider only the method of modification (3.10)' instead of (3.10), i.e., (3.21) can be replaced by

$$(3.21)' \quad \begin{cases} i = 1 & \text{if} \quad \dot{u}^{\alpha}(t) < -\epsilon_1 k_1 \\ i = 2 & \text{if} \quad \dot{u}^{\alpha}(t) > \epsilon_2 k_2 \end{cases}$$

for the vector $\{F_{\beta}^{ij}\}$.

The numerical scheme which is employed here has the following final form:

$$(3.26) \qquad u_k^{\alpha,n} = (1-\omega) u_{k-1}^{\alpha,n} - \omega \left(\sum_{\beta=1}^{\alpha-1} \hat{S}_{\alpha\beta}^{i} u_k^{\beta,n} + \sum_{\beta=\alpha+1}^{N} \hat{S}_{\alpha\beta}^{i} u_{k-1}^{\beta,n} - \hat{F}_{\beta}^{ij} \right) / \hat{S}_{\alpha\alpha}^{i}$$

(α no summation)

Here $u_k^{\alpha,n}$ is the value of u^{α} for the k-th iteration of the S.O.R. method at time $n\Delta t$, ω is the iteration factor, and

$$(3.27) \qquad \begin{cases} \hat{S}_{\alpha\beta}^{i} = M_{\alpha\beta}^{i}/\Delta t + \theta S_{\alpha\beta}^{i} \\ \hat{F}_{\beta}^{ij} = F_{\beta}^{ij} + L_{\beta}^{j} + M_{\alpha\beta}^{i} u^{\beta,n-1}/\Delta t - (1-\theta) S_{\alpha\beta}^{i} u^{\beta,n-1} \\ 0 \le \theta \le 1. \end{cases}$$

<u>EXAMPLE 6</u>. Figure 6-(a) shows the domain D and its boundary conditions. We employ rectangular linear isoparametric elements with $16 \times 16 = 256$ meshes whose size is 0.025. Material constants are

$k_1 = 1.0, \quad C_1 = 1.0 \quad$ for solid part $D_1(t)$,

$k_2 = 1.0, \quad C_2 = 0.5 \quad$ for melted part $D_2(t)$,

$\ell = 1.0.$

The initial temperature θ_0 is given by $+1^\circ C$ everywhere. Dirichlet boundaries are considered at two points on the top surface given by

$$\theta(x,t) = g(t) = \begin{cases} -1.0 & \text{if } 0 < t \leq 0.4 \text{ and } 0.5 < t \leq 0.6 \\ 0.5 & \text{if } 0.4 < t \leq 0.5 \end{cases}$$

On other parts of the boundary, the Neumann condition $\partial\theta = 0$ is assumed. The time interval Δt is 0.1. We use $\theta = 1.0$ in (3.27), i.e., the implicit θ-scheme of time discretization and $\omega = 1.4$ as the overrelaxation factor of S.O.R. method. The judgement of convergence is done if the relative tolerance $\sum_\alpha \left| u_k^{\alpha,n} - u_{k-1}^{\alpha,n} \right| / \sum_\alpha \left| u_k^{\alpha,n} \right|$ is less than 10^{-4}.

In Figure 6(b) we show the case of $\epsilon_1 = \epsilon_2 = 0$ (see (3.21)'). Until time step 3, the frozen front propagates monotonically with fairly smooth interface, since the cooling at the top surface is monotone. The step 4 has somehow unstable values in temperature θ, which has a fairly irregular shape of the frozen front. We think this is because the frozen area is going to vanish almost at this stage. In order to avoid this irregularity, we tried several cases changing ϵ_1 and ϵ_2, which are shown in Figure 6(c). If we compare the figures at time step 4, we notice that the shape of the remaining frozen part is strongly affected by the values of ϵ_1 and ϵ_2. However, fortunately, this frozen area does not affect appreciably the temperature field of the subsequent time step, since the values of the freezing index u hardly change by the modification (3.10)'(the field variable is u, not temperature θ). The selection of ϵ_1 and ϵ_2 depends on our numerical and experimental experience, but it seems to us that $\epsilon_1 = \epsilon_2 = 0.001$ is fairly proper upon observing Figure 6(c).

References

[1] Aguirre-Puente, J., and Fremond, M., "Frost Propagation in Wet Porous Media," in Lecture Notes in Math. 503, ed. P. Germain and B. Nayroles, Springer-Verlag, Berlin, 1976, pp. 137-147.

[2] Duvaut, G., "The Solution of a Two-phase Stefan Problem by a Variational Inequality," in Moving Boundary Problem in Heat Flow and Diffusion, ed. J.R. Ockendon and W.R. Hodgikins, Clarendon Press, Oxford, 1975, pp. 173-181.

[3] Fremond, M., "Frost Propagation in Porous Media," in Computational Methods in Nonlinear Mechanics, The University of Texas at Austin, 1974.

[4] Kikuchi, N., "A Class of the Stefan Problem Using Variational Inequalities," U.S. - Japan Seminar on Interdisciplinary Finite Element Analysis, Aug. 1978.

[5] Lyons, J.L., Sur Quelquer Questions d'Analyse, de Mécanique et de Contrôle Optimal, Les Presses de l'Universite de Montréal. Montréal, 1976.

[6] Rubinstein, L.I., The Stefan Problem, A.M.S. Translation Vol. 27, Providence, 1971.

[7] Yamaguchi, M., and Nogi, T., The Stefan Problem (in Japanese), Sangyo-Tosho, Tokyo, 1977.

TABLE

FIGURE

Table 1. Convergence Test for EXAMPLE 1.

Δt	$2k_1 \Delta t$	$\Delta \ell$	convergence
10	20	2	o.k.
5	10	2	o.k.
1	2	2	o.k.
0.5	1	2	o.k.
0.1	0.2	2	NO

$k_1 = 1.0$

$\Delta = 0.02$

$\ell = 100.$

DAY (t)	g(0,t)	g(L,t)
10	-7.0	9.5
20	-9.5	8.5
30	-12.5	7.5
40	-16.5	6.5
50	-19.0	5.5
60	-21.5	5.0
70	-24.0	5.0
80	-24.0	4.5
90	-21.5	3.5
100	-18.5	3.0
110	-16.0	3.0
120	-12.5	3.5
130	-7.5	4.5
140	-2.5	5.0
150	1.0	5.0
160	3.5	5.5
170	7.0	6.0
180	9.5	6.5
190	11.0	7.0
200	13.5	8.0

Table 2. Boundary Conditions for EXAMPLE 2

Table 3. Convergence Test for EXAMPLE 2

Δ	$k_1 \Delta t g(0, \Delta t)/2$	$\Delta^2 \ell$	convergence
0.1	1500.	240.	o.k.
0.2	1500.	960.	No Good
0.5	1500.	6000.	No

$k_1 = 60.$

$\ell = 24000.$

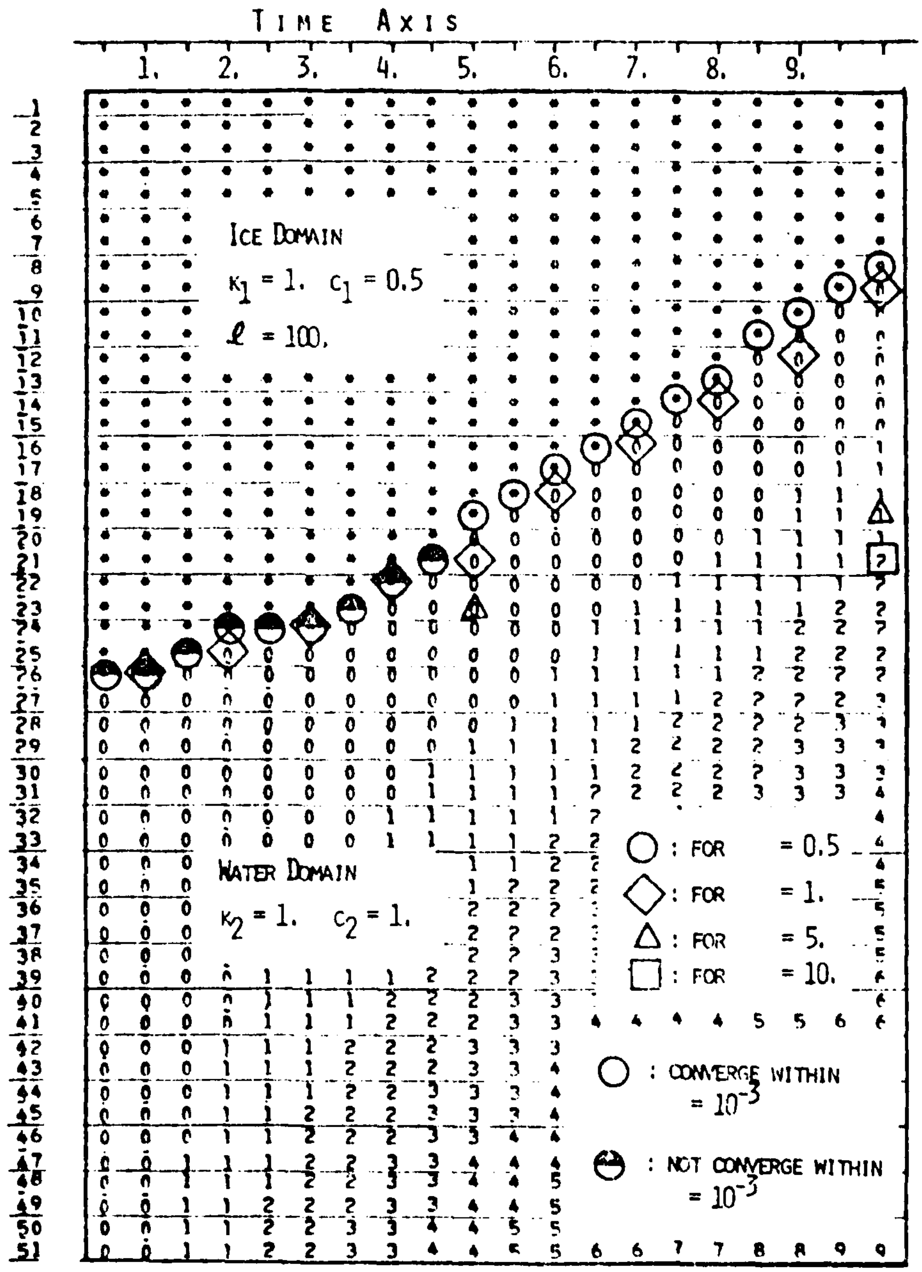


Figure 1. Results for EXAMPLE 1.

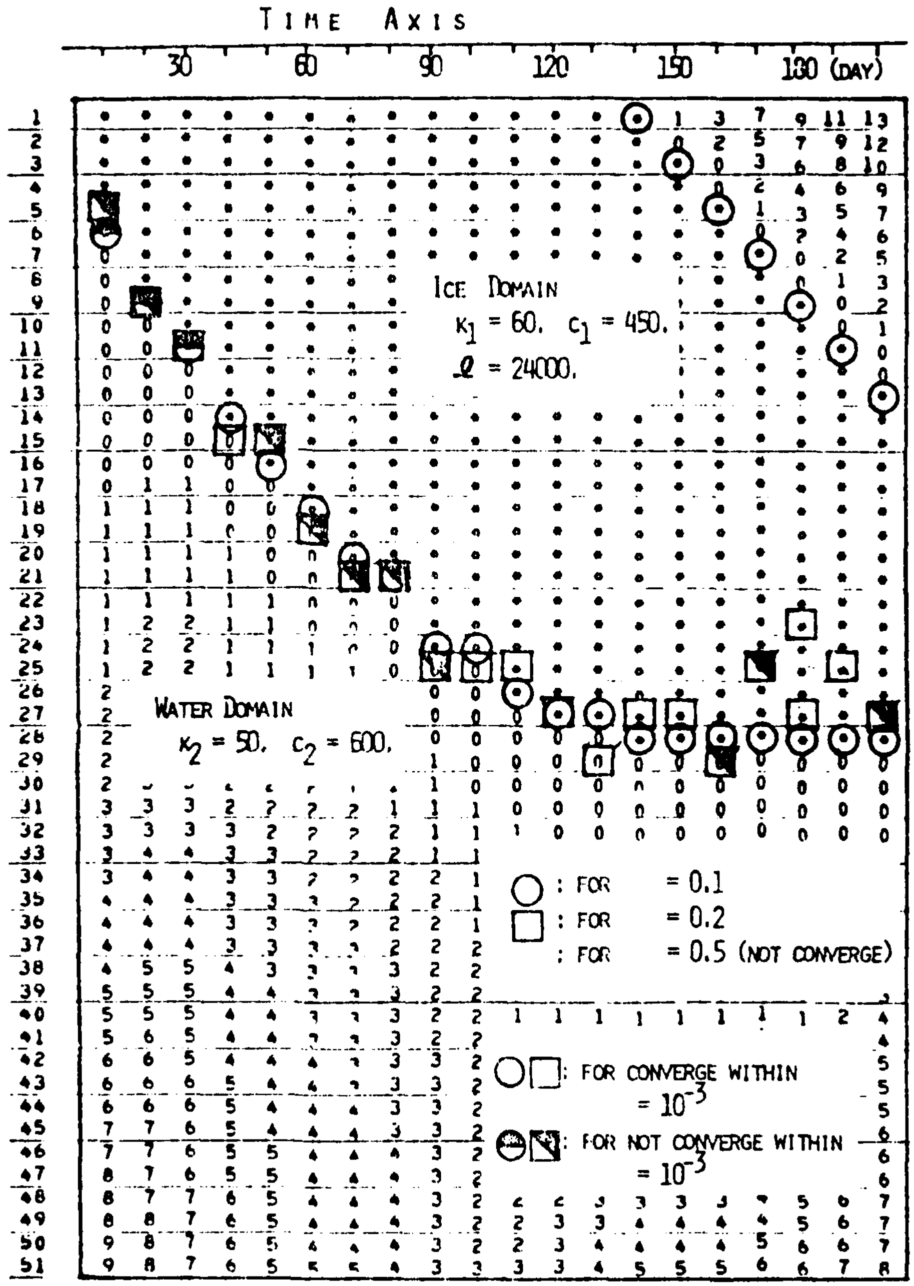


Figure 2. Results for EXAMPLE 2.

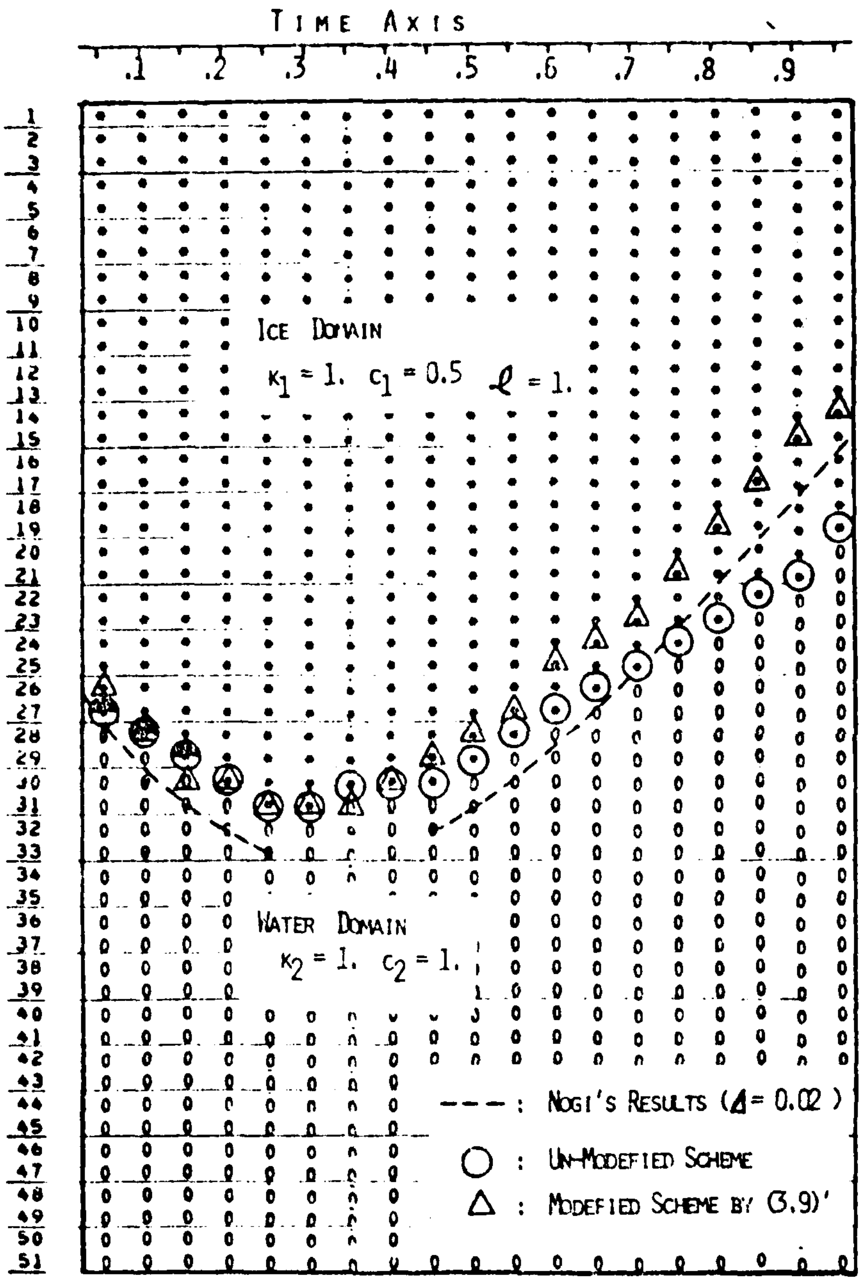
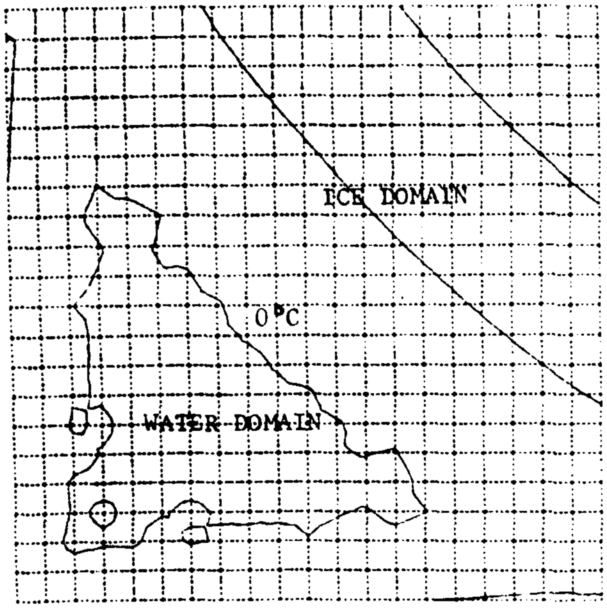
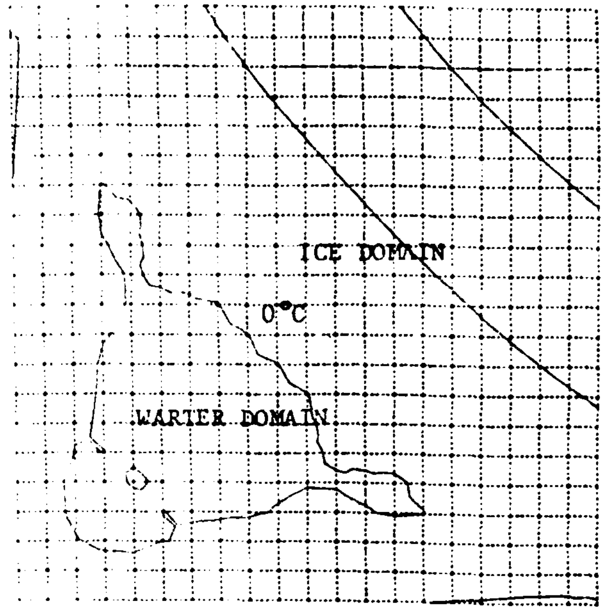


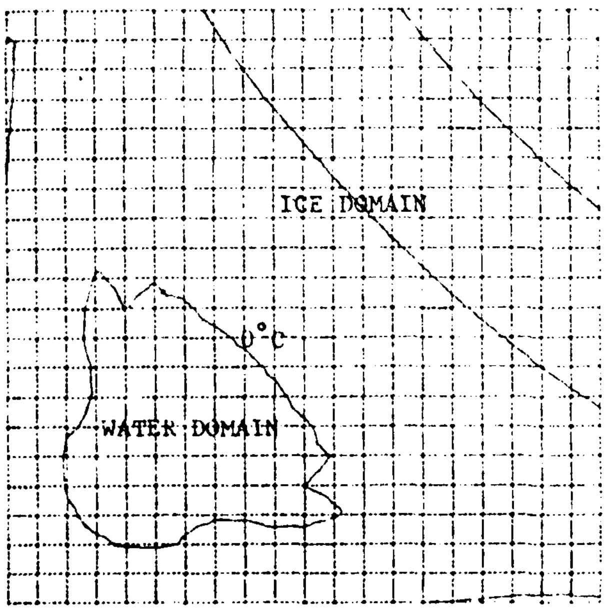
Figure 3. Results for EXAMPLE 3.



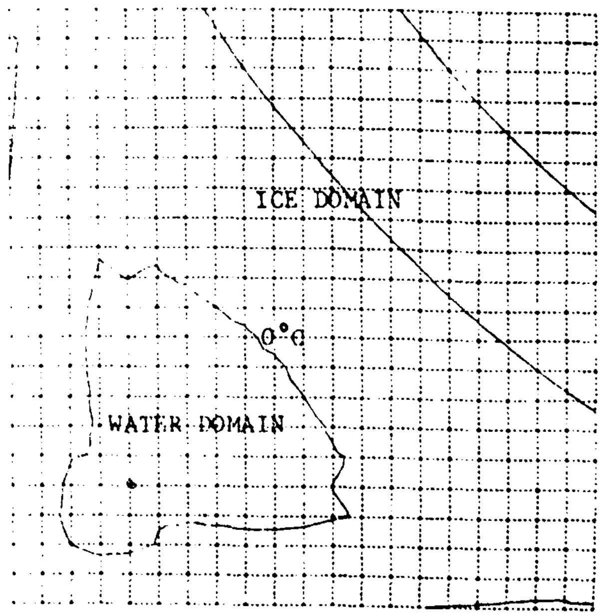
h_1=0. and h_2=1.



h_1=1. and h_2=4.



h_1=1. and h_2=1.



h_1=2. and h_2=1.

Figure 4-(a) Results for EXAMPLE 4 at t=0.40

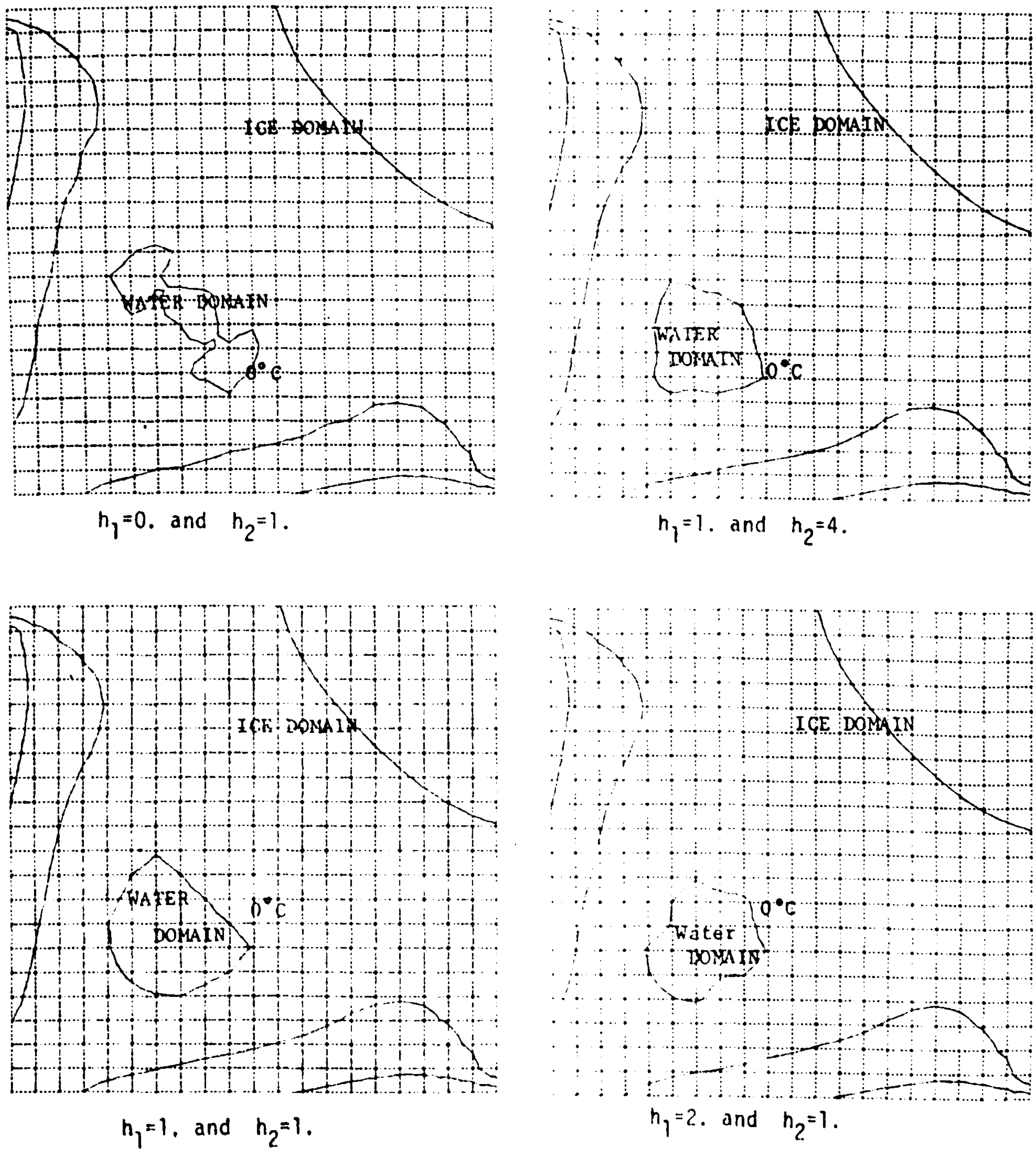


Figure 4-(b) Results for EXAMPLE 4 at t=0.45

FIGURE 5. RESULTS FOR EXAMPLE 5.

(A) : NO MODEFICATION

(B) : $\varepsilon_1 = \varepsilon_2 = 0.001$

(C) : $\varepsilon_1 = \varepsilon_2 = 0.01$

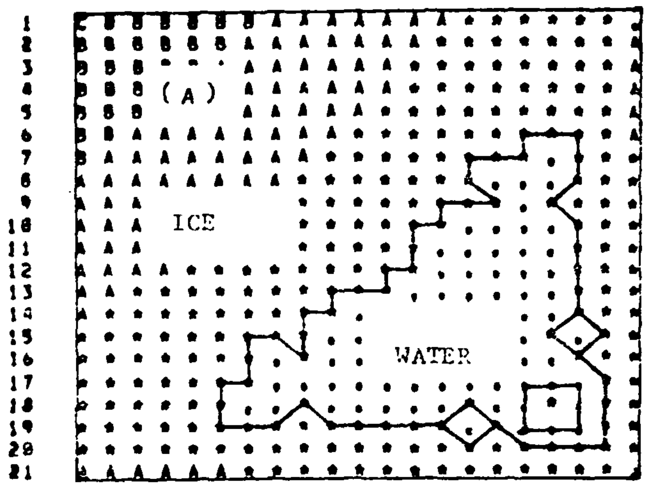
. : WATER

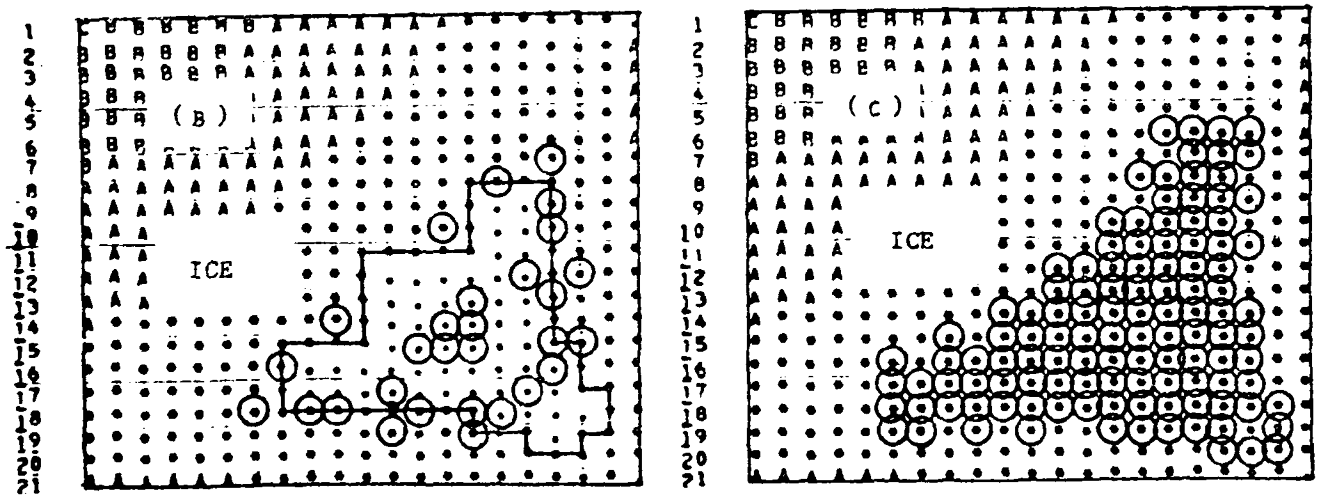
○ : TRANSIENT REGION

$c_1 = 0.5 \quad c_2 = k_1 = k_2 = 1.$

LATENT HEAT $\ell = 1.$

$\Delta = 0.02 \quad \Delta t = 0.05$





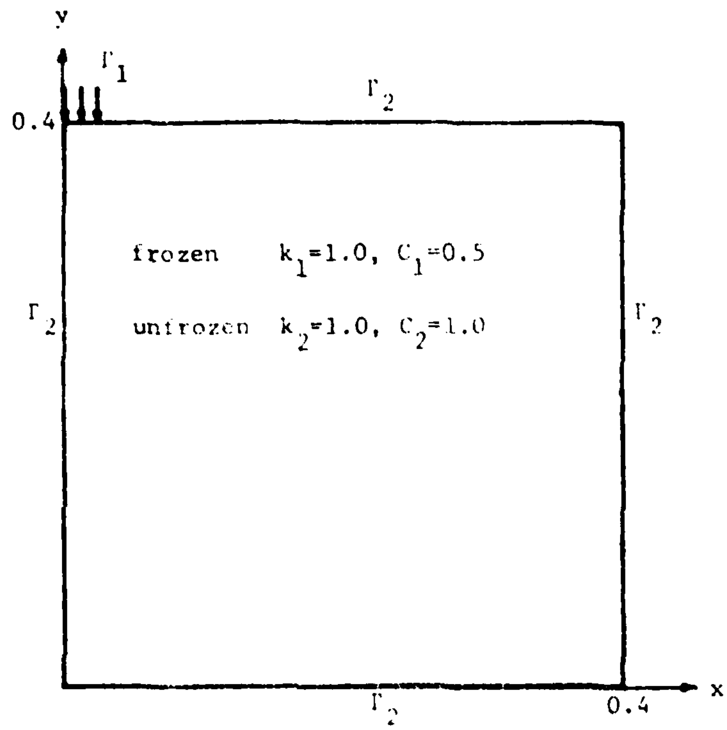


Figure 6-(a) Two dimensional model

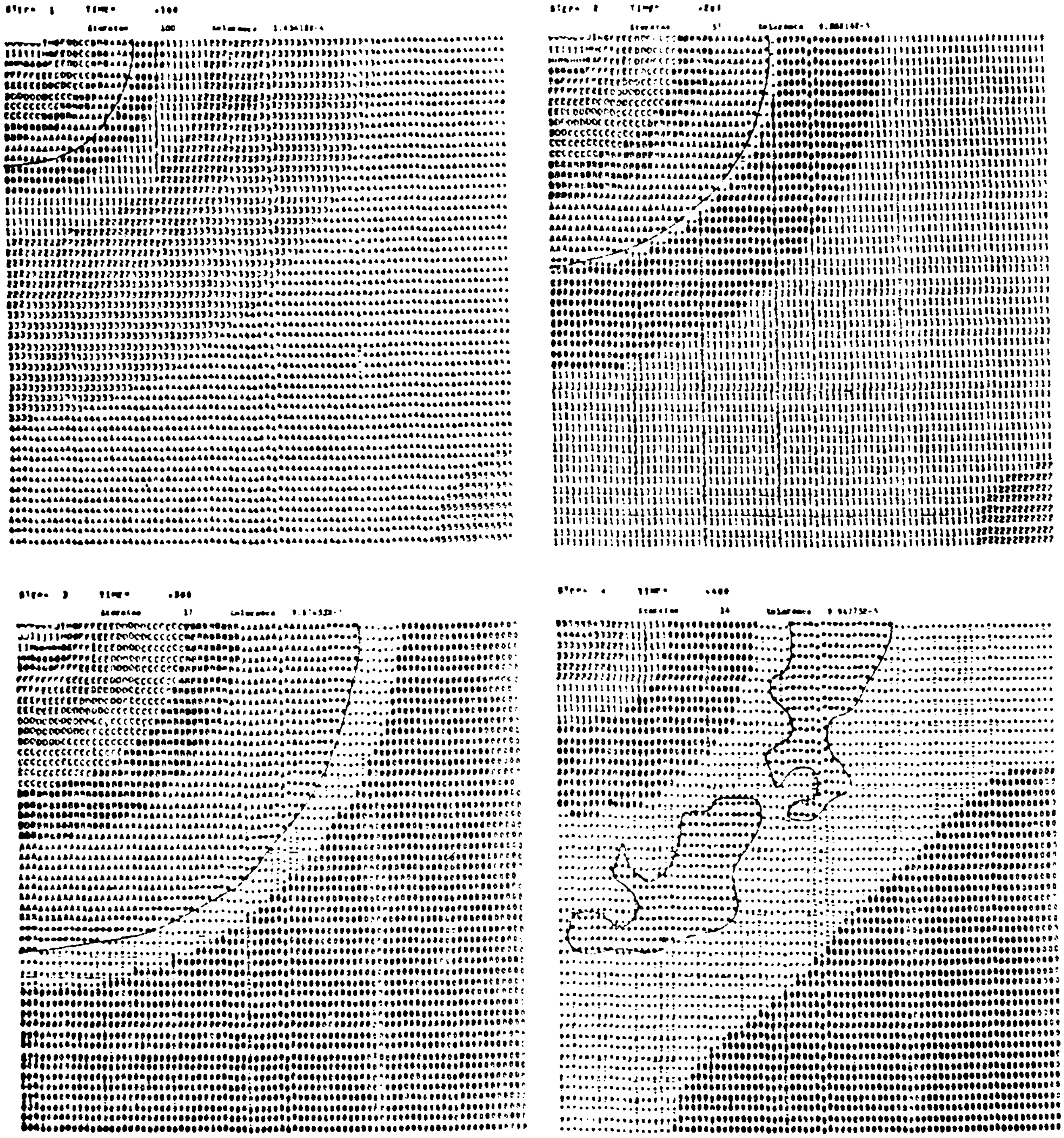


Figure 6-(b) Results for EXAMPLE 6
(Propagation of the Frost Line)

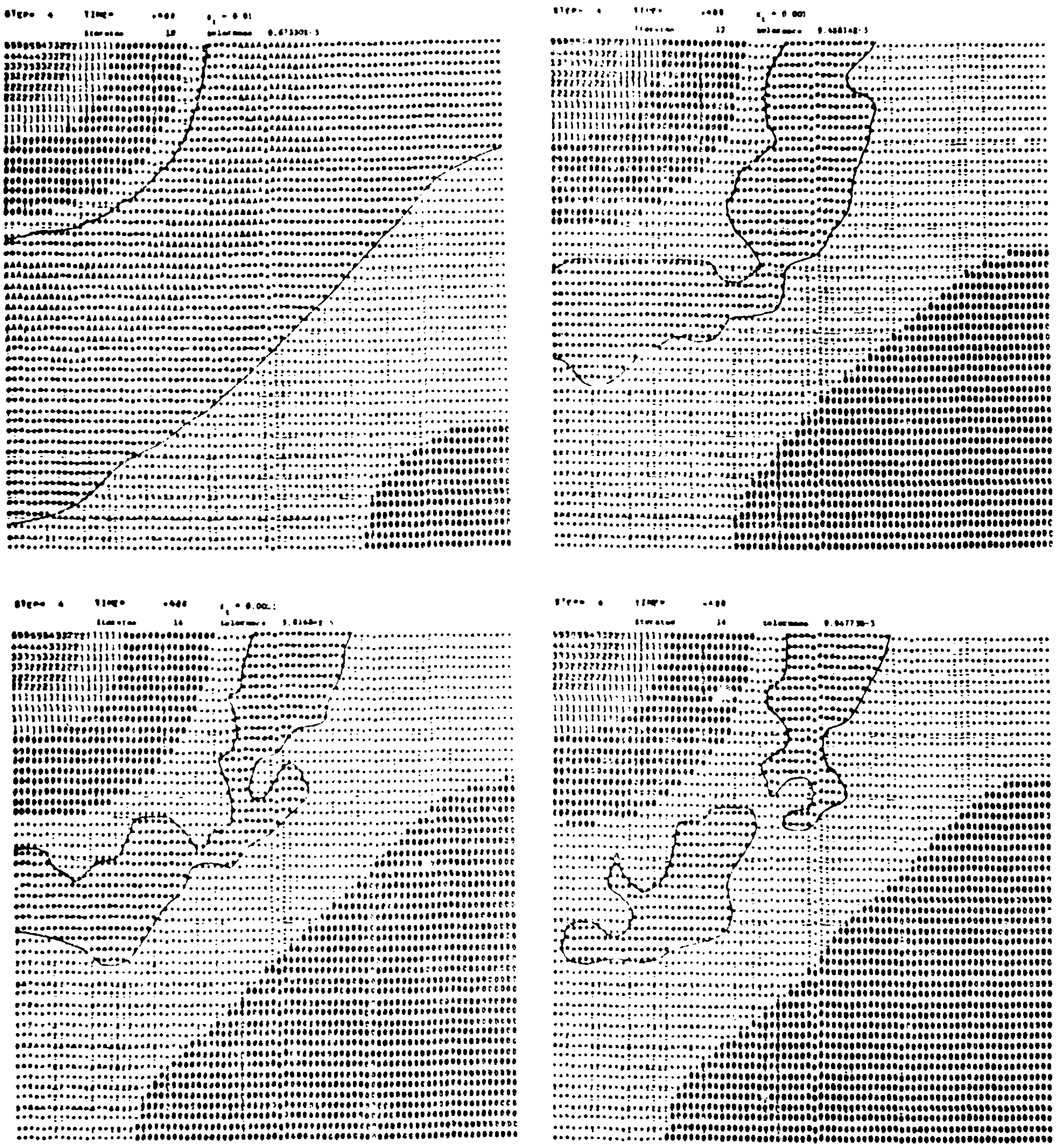


Figure 6-(c) Results for EXAMPLE 6
(Comparison of the Transient Region)

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APPROXIMATION AND NUMERICAL ANALYSIS OF NONLINEAR EQUATIONS OF --ETC(U)
JAN 80 J T ODEN DAAG29-77-G-0087

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APPENDIX C

Qualitative Analysis and Galerkin Approximations of a Class of Pseudomonotone Diffusion Problems

1. INTRODUCTION

In this paper, we present a qualitative analysis of solutions and Galerkin approximations of a class of nonlinear diffusion problems characterized by nonlinear parabolic equations of the type

$$\frac{\partial u}{\partial t} + A(u) = f$$

where u is an element of a separable reflexive Banach space $\mathcal{W}$ densely and continuously embedded in another Banach space $\mathcal{V}$ and A is a coercive $\mathcal{W}$-pseudomonotone operator from its domain $\mathcal{W}$ in $\mathcal{V}$ onto the dual space $\mathcal{V}'$. Since A is not monotone, the analysis of problems of this type is complicated by the possibility of non-unique solutions, an absence of continuous dependence on the data, and the corresponding absence of stability.

Most of our attention will be focused on the following class of problems: Find $u = u(x,t)$, $(x,t) \in \Omega \times (0,T)$, such that

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + A(u) &= f \ , \quad x \in \Omega \ , \quad 0 < t < T \\ u &= 0 \ , \quad x \in \partial\Omega, \quad 0 < t < T \\ u(x,0) &= u_o \ , \quad x \in \Omega \end{aligned}\right\} \qquad (1.1)$$

where

$$\begin{aligned} A(u) &= A_1(u) + A_2(u) \\ &= -\nabla \cdot \underset{\sim}{a}(x,\nabla u) + b(x,u,\nabla u) \ , \ x \in \Omega \end{aligned} \qquad (1.2)$$

$$\left.\begin{aligned} &\underset{\sim}{a}(x,\nabla u) = a(x)\nabla u + k(x)|\nabla u|^{p-2}\nabla u \\ &\quad a,k \in L^\infty(\Omega) \ , \ 2 \le p < \infty \\ &a(x) \ge a_o \ge 0 \ , \quad k(x) \ge k_o > 0 \quad \text{a.e. on } \Omega \end{aligned}\right\} \qquad (1.3)$$

and $b(\zeta,\underset{\sim}{\zeta}) = b(x,\zeta,\underset{\sim}{\zeta})$ is a totally Fréchet differentiable function in $\mathbb{R} \times \mathbb{R}^n$ for which there are positive reals q and r such that

$$\left.\begin{aligned} &|b(\zeta,\underset{\sim}{\zeta})| \le c|\zeta|^q|\underset{\sim}{\zeta}|^r \\ &|\partial_\zeta b(\zeta,\underset{\sim}{\zeta})| \le c_q|\zeta|^{q-1}|\underset{\sim}{\zeta}|^r \quad (q \ne 0) \\ &|\partial_{\underset{\sim}{\zeta}} b(\zeta,\underset{\sim}{\zeta})| \le c_r|\zeta|^q|\underset{\sim}{\zeta}|^{r-1} \quad (r \ne 0) \end{aligned}\right\} \qquad (1.4)$$

and

$$\left.\begin{array}{c} q = 0 \ \text{ or } \ q \geq 1 \ , \quad r = 0 \ \text{ or } \ r \geq 1 \\ 1 \leq q + r < p - 1 \end{array}\right\} \qquad (1.5)$$

Here Ω is an open bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, $0 < T < \infty$, f is given data in $\Omega \times (0,T)$, and u_o is initial data given on Ω .

The operator A_1 is a generalization of a part of a nonlinear heat diffusion operator proposed by Coleman and Mizel [4]. With $a \equiv 0$, it also corresponds, when spaces of vector functions with zero divergence are considered, to the diffusion part of the generalized Navier-Stokes equation studied by Lions in [10] and [11]. The general operator A_2 may model a convective part of the process. For example, convective terms such as those in the Navier-Stokes equations, or in heat conduction problems when convective terms due to chemical reactions are present. This type of diffusion equation is important beyond the study of thermomechanical phenomena; it also occurs in modeling biological, social and other phenomena (Cf. Fitzgibbon and Walker [6] for a survey of nonlinear diffusion models.) We should mention Tsutsumi's study, [16], where a non-monotone parabolic problem in which an equation of the form $\partial u/\partial t - \sum_{i=1}^{n} \partial(|\partial u/\partial x_i|^{p-2}\partial u/\partial x_i)/\partial x_i - u^q = 0$ is analyzed, which is a special case of the problem considered here.

This study is divided into two principal parts. Part I, Pseudomonotone Parabolic Problems, is devoted to the study of the existence of solutions of a general class of non-monotone, nonlinear parabolic equations. After some preliminaries on properties of certain function spaces are laid down in the section following this Introduction, a general class of nonlinear parabolic problems involving coercive pseudomonotone operators is given in Section 3 together with an existence theorem for such problems. An existence theorem for problems of this type has been given by Lions [11], but to lay groundwork for our study of Galerkin approximations taken up in Section 9 of the paper, we give an alternate proof. The principal tool, both in the analysis of the general problem and in the approximation of the model problem (1.1) - (1.5), is the construction of an elliptic regularization of the given problem. This regularization, of the type used by Lions [10], is introduced and analyzed in Section 4 and the proof of the existence theorem for the pseudomonotone parabolic problem is completed in Section 5. In Section 6 of the paper, we give a brief summary of some results of Oden [13] which provide sufficient conditions for pseudomonotonicity of nonlinear operators on reflexive Banach spaces.

Part II of this study is concerned with the specific class of nonlinear diffusion problems characterized by (1.1) - (1.5). In Section 7, we show that the operator A defined in (1.2) is coercive and pseudomonotone on a dense continuously embedded subspace of the Banach space $L^p(0,T;\ W_o^{1,p}(\Omega))$ and that solutions to (1.1) do exist in $L^\infty(0,T;\ L^2(\Omega)) \bigcap L^p(0,T;\ W_o^{1,p}(\Omega))$ under the conditions (1.3) - (1.5). In general, multiple solutions will exist to (1.1) and there cannot exist a continuous dependence on the data. However, regularity conditions on the solutions can be given which will guarantee their uniqueness, and these are discussed in Section 8.

Sections 9 and 10 are devoted to studies of Galerkin approximations of the model problem (1.1) - (1.5). In Section 9, we describe properties of space - time Galerkin approximations of solutions in $L^p(0,T;\ W_o^{1,p}(\Omega))$ and we give an approximation theorem which establishes their strong convergence (in $L^p(0,T;\ W_o^{1,p}(\Omega))$). We also derive error estimates for such approximations. Finally, in Section 10, we describe Faedo-Galerkin (semi-discrete) approximations. We note that, in general, this type of semi-discrete approximation is not necessarily well-defined for coercive pseudomonotone parabolic problems. However, in the case of our model problem, it is proved in Theorem 10.1 that sufficient conditions are satisfied which guarantee existence and also <u>uniqueness</u> of Faedo-Galerkin approximations to problem (1.1) - (1.5). We also prove sufficient conditions for weak and strong convergence of such approximations and we establish corresponding approximation error estimates.

PART I. PSEUDOMONOTONE PARABOLIC PROBLEMS

2. Some Preliminaries

The following notations and conventions will be in force throughout this study:

$(V, \|\cdot\|)$ = a real, separable, reflexive Banach space.

$(V', \|\cdot\|_*)$ = the dual space of V.

$\langle\cdot,\cdot\rangle$ = duality pairing on $V' \times V$; i.e., for $v' \in V'$ and $v \in V$, $\langle v',v\rangle = v'(v)$.

$(H,(\cdot,\cdot),|\cdot|)$ = a real Hilbert space identified with its dual, in which V is densely and continuously embedded: $V \hookrightarrow H = H'$. Then H is a pivot space such that

$$V \hookrightarrow H \hookrightarrow V' \tag{2.1}$$

Next, denoting by $t \in [0,T]$, $0 < T < \infty$, the time variable, we introduce the space of vector functions of time

$$\begin{aligned}(\mathcal{V}, |||\cdot|||) &= L^p(0,T;V) \\ &= \left\{ v\colon [0,T] \to V \; ; \; |||v||| = \left(\int_0^T \|v(t)\|^p \, dt\right)^{1/p} < \infty \right\} \quad 2 \le p < \infty \end{aligned} \tag{2.2}$$

which is a separable, reflexive Banach space, whose dual space can be identified as

$$(\mathcal{V}', |||\cdot|||_*) = L^{p'}(0,T;V') \; , \; p' = p/(p-1) \; .$$

$[\cdot,\cdot]$ = duality pairing on $\mathcal{V}' \times \mathcal{V}$, i.e., for $v' \in \mathcal{V}'$ and $v \in \mathcal{V}$,

$$[v',v] = \int_0^T \langle v'(t),v(t)\rangle \, dt \tag{2.3}$$

$(\mathcal{H},(\cdot,\cdot)_{\mathcal{H}},|\cdot|_{\mathcal{H}}) = L^2(0,T;H)$ equipped with the natural inner product and norm, which being identical with its dual, is a pivot Hilbert space such that

$$\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}' \tag{2.4}$$

$\mathcal{D}((0,T))$ = space of test functions defined on $(0,T)$; i.e.,

$\phi \in \mathcal{D}((0,T)) \Leftrightarrow \phi \in C_o^\infty((0,T))$ with the usual locally convex linear topological structure.

$\mathcal{D}'((0,T);X) = L(\mathcal{D}((0,T)),X) =$ the space of distributions on $\mathcal{D}((0,T))$ with values in some normed linear space X .

We observe that since $V \subset \mathcal{D}'((0,T);V)$, every $v \in V$ defines a distribution on $\mathcal{D}((0,T))$, also denoted by V , with values in v , given by

$$v(\phi) = \int_0^T v(t)\phi(t)dt \ , \quad \forall \quad \phi \in \mathcal{D}((0,T))$$

whose distributional time derivatives, also belonging to $\mathcal{D}'((0,T);V)$, are defined by

$$\frac{\partial^m v}{\partial t^m}(\phi) = (-1)^m \int_0^T v(t)\,\frac{d^m\phi(t)}{dt^m}\,dt \ , \quad \forall \quad \phi \in \mathcal{D}((0,T))$$

Finally, we introduce the separable, reflexive Banach spaces $(\mathcal{U}, |||\cdot|||_{\mathcal{U}})$ and $(\mathcal{W}, |||\cdot|||_{\mathcal{W}})$ defined by

$$\left.\begin{aligned} &\mathcal{U} = \{v: \ v \in V \ , \ \dot{v} = \partial v/\partial t \in H\} \\ &|||v|||_{\mathcal{U}} = |||v||| + |\dot{v}|_H \end{aligned}\right\} \qquad (2.5)$$

$$\left.\begin{aligned} &\mathcal{W} = \{v: \ v \in V \ , \ v = \partial v/\partial t \in V'\} \\ &|||v|||_{\mathcal{W}} = |||v||| + |||\dot{v}|||_* \end{aligned}\right\} \qquad (2.6)$$

which satisfy the relation (with dense inclusions and continuous injections)

$$\mathcal{U} \hookrightarrow \mathcal{W} \hookrightarrow V \qquad (2.7)$$

and whose elements possess the following properties (cf. [8] and [10]):

i) $\mathcal{W}$ is continuously embedded in $C([0,T];H)$; i.e., if $v \in \mathcal{W}$, then, after an eventual modification on a set of measure zero in $[0,T]$, v is continuous from $[0,T]$ into H and there is a constant K , independent of v , such that

$$\sup_{t \in [0,T]} |v(t)| \leq K\,|||v|||_{\mathcal{W}} \qquad (2.8)$$

ii) If $u,v \in \mathcal{W}$, then u,v satisfy the Green's formula

$$[\dot{u},v] = (u(T),v(T)) - (u(0),v(0)) - [\dot{v},u] \qquad (2.9)$$

iii) The trace mappings $v \mapsto v(0)$ and $v \mapsto v(T)$ from $\mathcal{W} \to H$ are surjective and their restrictions to $\mathcal{U}$ have range dense in H ; i.e.,

$$\{v(0): v \in \mathcal{W}\} = H = \{v(T): v \in \mathcal{W}\} \tag{2.10}$$

$$\{v(0): v \in \mathcal{U}\} \text{ and } \{v(T): v \in \mathcal{U}\} \text{ are dense in } H \tag{2.11}$$

We also remark that we make frequent use of Young's inequality in subsequent analyses: If $x,y \in \mathbb{R}$, $1 < s < \infty$, $s' = s/(s-1)$, and b is any real > 0, then

$$xy \leq \frac{b^s}{s} |x|^s + \frac{1}{s' b^{s'}} |y|^{s'} \tag{2.12}$$

3. Existence Theorem

With the conventions of the previous section in force, we now consider a general class of non-monotone evolution problems characterized as follows:

Given $f \in \mathcal{V}'$ and $u_o \in H$, find $u \in \mathcal{W}$ such that

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} + A(u) = f \\ &u(0) = u_o \end{aligned}\right\} \tag{3.1}$$

Here A is an operator from $\mathcal{V}$ into $\mathcal{V}'$, possibly depending upon the time parameter $t \in (0,T)$, satisfying the following conditions:

AI. $A: \mathcal{V} \to \mathcal{V}'$ is $\mathcal{W}$-pseudomonotone*, i.e.,

i) A is bounded in the sense that it maps bounded sets in $\mathcal{V}$ into bounded sets in $\mathcal{V}'$.

ii) If $\{u_n\} \subset \mathcal{W}$ is a sequence converging weakly to $u \in \mathcal{W}$ and if

$$\limsup_{n\to\infty} [A(u_n) , u_n - u] \leq 0$$

then, $\forall \; v \in \mathcal{V}$,

$$\liminf_{n\to\infty} [A(u_n) , u_n - v] \geq [A(u) , u-v]$$

AII. $A: \mathcal{V} \to \mathcal{V}'$ is coercive, i.e.,

$$\frac{[A(v) , v]}{|||v|||} \to +\infty \text{ as } |||v||| \to \infty$$

*We will refer to $A: \mathcal{V} \to \mathcal{V}'$ as pseudomonotone if it is $\mathcal{V}$-pseudomonotone.

<u>Theorem 3.1</u>. Let the operator $A\colon \mathcal{V} \to \mathcal{V}'$ satisfy conditions (AI) and (AII). Then there exists at least one solution $u \in \mathcal{W}$ to problem (3.1). $\square$

A proof of this theorem was given by Lions [11, Chap. 3] which makes use of the method of elliptic regularization. Lions' method, which was introduced in his study of linear parabolic problems [9], effectively involves converting (3.1) into an elliptic problem, depending on a real parameter $\varepsilon > 0$, and constructing solutions to (3.1) as limiting cases when $\varepsilon \to 0^+$. When A is a spatial differential operator, the method described in Lions [11] leads to an integro-differential equation involving ε.

However, one of the principal objectives of the present work is to study properties of Galerkin approximations of (3.1) and, in particular, to obtain a priori estimates for such approximations. The general method employed by Lions does not lead to results from which a constructive approximation theory can be easily established. For this reason, we give here an alternative proof of Theorem 3.1. which was suggested by Lions [11], also based on the notion of elliptic regularization, but in which a different form of the regularized problem is used. We will show in Section 9 that the constructive nature of our proof is useful in studies of Galerkin approximations. Our method generalizes those used by Lions in his study of nonlinear parabolic problems [10] and by Dubinskii in the analysis of parabolic problems with semibounded variation [5].

4. An Elliptic Regularization

We begin by recalling that $\mathcal{U}$ denotes the separable, reflexive Banach space defined by (2.5) which is everywhere dense in $\mathcal{W}$. We will denote by $[\cdot,\cdot]_{\mathcal{U}}$ duality pairing on $\mathcal{U}' \times \mathcal{U}$.

We next introduce a family of "elliptic" operators $A_\varepsilon\colon \mathcal{U} \to \mathcal{U}'$, ε is a positive real number, defined by

$$\begin{aligned}[A_\varepsilon(u),v]_{\mathcal{U}} &= \varepsilon\left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)_H - \left(u,\frac{\partial v}{\partial t}\right)_H + (u(T),v(T)) \\ &\quad + [A(u),v] \; ; \quad u,v \in \mathcal{U} \end{aligned} \tag{4.1}$$

where A is the operator in (3.1) which satisfies conditions (AI) and (AII). The problem of finding $u_\varepsilon \in \mathcal{U}$ such that

$$[A_\varepsilon(u_\varepsilon),v]_{\mathcal{U}} = [f,v] + (u_o,v(0)) \quad \forall \quad v \in \mathcal{U} \tag{4.2}$$

is an elliptic regularization of (3.1) obtained (formally) by adding to $\partial u/\partial t + A(u)$ the term $-\varepsilon\partial^2 u/\partial t^2$. We will first show that (4.2) is solvable and then prove that solutions to (3.1) are obtained as $\varepsilon \to 0^+$.

Lemma 4.1 The operator $A_\varepsilon: \mathcal{U} \to \mathcal{U}'$ defined by (4.1) is (i) $\mathcal{U}$-pseudomonotone and (ii) coercive.

Proof. (i) We observe that the operator $A_\varepsilon: \mathcal{U} \to \mathcal{U}'$ can be expressed by the sum $A_\varepsilon = B_\varepsilon + A$ where B_ε is the linear operator on $\mathcal{U}$ defined by

$$[B_\varepsilon u,v]_{\mathcal{U}} = \varepsilon\left(\frac{\partial u}{\partial t},\frac{\partial v}{\partial t}\right)_H - \left(u,\frac{\partial v}{\partial t}\right)_H$$

$$+ (u(T),v(T)) \ ; \quad u,v \in \mathcal{U} \tag{4.3}$$

We prove first that $B_\varepsilon: \mathcal{U} \to \mathcal{U}'$ is positive (monotone) and continuous. Indeed,

$$[B_\varepsilon u,u]_{\mathcal{U}} = \varepsilon|\dot{u}|_H^2 + \frac{1}{2}|u(0)|^2 + \frac{1}{2}|u(T)|^2 \geq 0$$

and

$$\begin{aligned}[B_\varepsilon u,v]_{\mathcal{U}} &= (\dot{u},\dot{v})_H + (\dot{u},v)_H + (u(0),v(0)) \\ &\leq \varepsilon|\dot{u}|_H\,|\dot{v}|_H + |\dot{u}|_H\,|v|_H + |u(0)|\,|v(0)| \\ &\leq (\varepsilon + k_1 + k_2^2)\,|||u|||_{\mathcal{U}}\,|||v|||_{\mathcal{U}}\end{aligned}$$

where k_1 and k_2 denote the continuous embedding constants of $V \hookrightarrow H$ and $\mathcal{U} \hookrightarrow C([0,T];H)$, respectively.

Next note that because of condition (AI), A is $\mathcal{U}$-pseudomonotone as an operator from $\mathcal{U}$ into $\mathcal{U}'$. Hence the operator A_ε is pseudomonotone since it is the sum of a continuous monotone linear operator and a pseudomonotone operator (cf. [11, p. 189]).

(ii)
$$[A_\varepsilon(v),v]_{\mathcal{U}} = \varepsilon|\dot{v}|_H^2 + \frac{1}{2}|v(0)|^2 + \frac{1}{2}|v(T)|^2$$

$$+ [A(v),v]$$

Hence
$$\frac{[A_\varepsilon(v),v]_U}{|||v|||_U} = \frac{[A(v),v]+\varepsilon|\dot{v}|_H^2}{|||v||| + |\dot{v}|_H} + \frac{1}{2}\,\frac{|v(0)|^2+|v(T)|^2}{|||v|||_U}$$

Since A is coercive on V by assumption (AII), the first term in the last result $\to +\infty$ as $|||v|||_U \to \infty$ and this proves the coercivity of A_ε on U. □

Therefore, since Lemma 4.1 establishes sufficient conditions for the surjectivity of $A_\varepsilon: U \to U'$ (cf. [11]), the following existence theorem holds.

<u>Theorem 4.1.</u> For any fixed $\varepsilon > 0$, there exists at least one solution $u_\varepsilon \in U$ of problem (4.2). □

<u>5. Proof of Existence Theorem 3.1</u>

We now return to the existence theorem (Theorem 3.1) for the nonmonotone parabolic equation (3.1) in which the operator $A: V \to V'$ satisfies conditions (AI) and (AII). Up to this point, we have shown that the elliptic regularization (4.2) has a solution in U for any $\varepsilon > 0$. We now show that solutions to (3.1) are obtained as $\varepsilon \to 0^+$. This is accomplished by following the standard procedure: (a) establishment of a priori bounds using the boundedness and coercivity of A; (b) passage to the limit as $\varepsilon \to 0^+$; (c) use of pseudomonotonicity arguments.

<u>Lemma 5.1.</u> Let $u_\varepsilon \in U$ be a solution of (4.2). Then, for every $\varepsilon > 0$, there exist positive constants C_1, C_2, C_3, and C_4, independent of ε, such that

$$\left.\begin{array}{l} \text{i) } |||u_\varepsilon|||_W \le C_1 \text{ ; i.e., } |||u_\varepsilon||| \le C_1 \text{ and } \left|\left|\left|\dfrac{\partial u_\varepsilon}{\partial t}\right|\right|\right|_* \le C_1 \\ \text{ii) } \sqrt{\varepsilon}\left|\dfrac{\partial u_\varepsilon}{\partial t}\right|_H \le C_2 \\ \text{iii) } |u_\varepsilon(0)| \le C_3 \text{ and } |u_\varepsilon(T)| \le C_4 \end{array}\right\} \quad (5.1)$$

<u>Proof.</u> The first inequality in (i) and inequalities (ii) and (iii) follow from the boundedness and coercivity of A. Indeed, replacing v in (4.2) by u_ε, we have

$$\varepsilon|\dot{u}_\varepsilon|_H^2 + \frac{1}{2}|u_\varepsilon(0)|^2 + \frac{1}{2}|u_\varepsilon(T)|^2 + [A(u_\varepsilon),u_\varepsilon]$$
$$= [f,u_\varepsilon] + (u_o,u_\varepsilon(0))$$
$$\leq |||f|||_* \, |||u_\varepsilon||| + \frac{b^2}{2}|u_o|^2 + \frac{1}{2b^2}|u_\varepsilon(0)|^2$$

where b denotes the Young's constant. Setting $b = 1$ produces the inequality

$$\frac{[A(u_\varepsilon),u_\varepsilon]}{|||u_\varepsilon|||} \leq |||f|||_* + \frac{1}{2}\frac{|u_o|^2}{|||u_\varepsilon|||} \Longrightarrow |||u_\varepsilon||| \leq C_1$$

and by setting $b = \sqrt{2}$ the bounds (ii) and (iii) in (5.1) follow from

$$\varepsilon|\dot{u}_\varepsilon|_H^2 + \frac{1}{4}|u_\varepsilon(0)|^2 + \frac{1}{2}|u_\varepsilon(T)|^2$$
$$\leq \left(|||A(u_\varepsilon)|||_* + |||f|||_*\right)|||u_\varepsilon||| + |u_o|^2$$

To prove the remaining bound in (i), we identify a solution u_ε of (4.2) with a distribution $u_\varepsilon \in \mathcal{D}'(0,T):V)$ in the sense that u_ε satisfies the distributional equation

$$-\varepsilon\ddot{u}_\varepsilon + \dot{u}_\varepsilon = f - A(u_\varepsilon) \tag{5.2}$$

Then, proceeding as in [10], it follows that in H

$$\left.\begin{aligned} -\varepsilon\dot{u}_\varepsilon(0) + u_\varepsilon(0) &= u_o \\ \dot{u}_\varepsilon(T) &= 0 \end{aligned}\right\} \tag{5.3}$$

and, by integrating (5.2),

$$\dot{u}_\varepsilon(t) = \int_0^T \frac{1}{\varepsilon} e^{(t-\tau)/\varepsilon}[f(\tau) - A(\tau,u_\varepsilon(\tau))]d\tau$$

from which

$$|||\dot{u}_\varepsilon|||_* \leq \left\|\frac{1}{\varepsilon}e^{\tau/\varepsilon}\right\|_{L^1(-T,0)} |||f - A(u_\varepsilon)|||_*$$
$$\leq |||f - A(u_\varepsilon)|||_*$$

This, since A is bounded, completes the proof. □

Lemma 5.2. Let $\{u_\varepsilon\} \subset \mathcal{U}$ be a sequence of solutions to problem (4.2) obtained as $\varepsilon \to 0^+$. Then there exists a subsequence, also denoted $\{u_\varepsilon\}$, and functions $u \in \mathcal{W}$ and $\chi \in \mathcal{V}'$ such that, as $\varepsilon \to 0^+$,

$$
\begin{array}{llll}
\text{i)} & u_\varepsilon \rightharpoonup u & \text{weakly in } \mathcal{W}\text{; in fact,} \\
 & u_\varepsilon \rightharpoonup u & \text{weakly in } \mathcal{V} \text{ and} \\
 & \dfrac{\partial u_\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} & \text{weakly in } \mathcal{V}' \\
\text{ii)} & \varepsilon \dfrac{\partial u_\varepsilon}{\partial t} \rightharpoonup 0 & \text{weakly in } \mathcal{H} \\
\text{iii)} & A(u_\varepsilon) \rightharpoonup \chi & \text{weakly in } \mathcal{V}' \\
\text{iv)} & u_\varepsilon(0) \rightharpoonup u(0) & \text{weakly in H} \\
 & u_\varepsilon(T) \rightharpoonup u(T) & \text{weakly in H}
\end{array}
$$

<u>Proof.</u> The convergence results (i), (iii) and (iv) follow from Lemma 5.1 the fact that the Banach spaces $\mathcal{W}$, $\mathcal{V}$ and H are reflexive, and the boundedness of A. To establish (ii), note that from Lemma 5.1 $\sqrt{\varepsilon}\,|\dot{u}_\varepsilon|_{\mathcal{H}} \le C_2$, $\varepsilon > 0$. Hence, $\forall\ w \in \mathcal{H}$,

$$\varepsilon(\dot{u}_\varepsilon, w)_{\mathcal{H}} \le \sqrt{\varepsilon}\, C_2\, |w|_{\mathcal{H}} \to 0 \quad \text{as} \quad \varepsilon \to 0^+ \qquad \square$$

Therefore, by virtue of Lemma 5.2 and Green's formula (2.9), we see that in the limit as $\varepsilon \to 0^+$ equation (4.2) becomes

$$[\dot{u},v] + [\chi,v] = [f,v] + (u_o - u(0), v(0)) \quad \forall\ v \in \mathcal{U}$$

Then, using the fact that $\mathcal{U}$ is dense in $\mathcal{V}$ and property (2.11), we conclude that the limit u of u_ε satisfies the following equation:

$$\left.\begin{aligned} \left[\frac{\partial u}{\partial t}, v\right] + [\chi,v] &= [f,v] \quad \forall\ v \in \mathcal{V} \\ u(0) &= u_o \quad \text{in H} \end{aligned}\right\} \tag{5.4}$$

It remains to be shown that $\chi = A(u)$ in $\mathcal{V}'$. Toward this end, we observe that from (4.2), Lemma 5.2 and (5.4)

$$\lim_{\varepsilon \downarrow 0} \{\varepsilon|\dot{u}_\varepsilon|^2_{\mathcal{H}} - [\dot{u}_\varepsilon, u_\varepsilon] + (u_\varepsilon(T), u_\varepsilon(T)) + [A(u_\varepsilon), u_\varepsilon]\}$$

$$= [f,u] + (u_o, u(0))$$

$$= \lim_{\varepsilon \downarrow 0} \{-[\dot{u}, u_\varepsilon] + (u_\varepsilon(T), u(T)) + [A(u_\varepsilon), u]\}$$

from which it follows that

$$\lim_{\varepsilon\downarrow 0} [A(u_\varepsilon),u_\varepsilon-u] = -\lim_{\varepsilon\downarrow 0} \Big\{ \varepsilon|\dot{u}_\varepsilon|_H^2 + \frac{1}{2}|u_\varepsilon(0)-u(0)|^2 + \frac{1}{2}|u_\varepsilon(T)-u(T)|^2 \Big\}$$
$$\leq 0$$

Hence, since $A: \mathcal{V} \to \mathcal{V}'$ is $\mathcal{W}$-pseudomonotone,

$$\liminf_{\varepsilon\downarrow 0} [A(u_\varepsilon),u_\varepsilon-v] \geq [A(u),u-v] \quad \forall \quad v \in \mathcal{V}$$

and

$$\liminf_{\varepsilon\downarrow 0} [A(u_\varepsilon),u_\varepsilon-v] \leq \limsup_{\varepsilon\downarrow 0} [A(u_\varepsilon),u_\varepsilon] - [\chi,v]$$
$$\leq \limsup_{\varepsilon\downarrow 0} [A(u_\varepsilon),u] - [\chi,v]$$
$$= [\chi,u-v]$$

Consequently,

$$[A(u)-\chi,u-v] \leq 0 \quad \forall \quad v \in \mathcal{V}$$

and we conclude that

$$\chi = A(u) \quad \text{in} \quad \mathcal{V}' \tag{5.5}$$

The proof of Theorem 3.1 now follows immediately from (5.4) and (5.5).

6. Some Sufficient Conditions for Pseudomonotonicity

We will review briefly here some results of Oden [13] which provide useful tests for pseudomonotonicity of a certain class of operators. We first state a corollary of Aubin's compactness theorem [2].

Theorem 6.1. Let V_1 be a Banach space in which V is continuously embedded and consider the Banach space $\mathcal{Y}$ defined by

$$\left.\begin{aligned} &\mathcal{Y} = \Big\{ v: v \in \mathcal{V}, \ \dot{v} = \frac{\partial v}{\partial t} \in L^{p_1}(0,T;V_1), \ 1 < p_1 \leq \infty \Big\} \\ &|||v|||_{\mathcal{Y}} = |||v||| + \|\dot{v}\|_{L^{p_1}(0,T;V_1)}, \quad v \in \mathcal{Y} \end{aligned}\right\} \tag{6.1}$$

If X is a Banach space continuously embedded in V_1 and in which V is compactly embedded:

$$V \overset{c}{\hookrightarrow} X \hookrightarrow V_1 \tag{6.2}$$

then the injection from $\mathcal{V}$ into the space $L^p(0,T;X)$ is compact:

$$\mathcal{V} \overset{c}{\hookrightarrow} L^p(0,T;X) \tag{6.3}$$

$\square$

Following Oden [13], let $A(u)$ denote values of an operator A from the Banach space V into its dual V' which has the property that there exists a map $(u,v) \mapsto A(u,v)$, $t \in (0,T)$, from $V \times V$ into V' such that $A(u,u) = A(u)$ and the following conditions hold:

BI. $\forall \; v \in V$, $u \mapsto A(u,v)$ is hemicontinuous from V into V' ; i.e., $\forall \; u,v,w \in V$, the function

$$\phi(s) = [A(u+sw,v),w] \;,\; s \in R$$

is continuous in s .

BII. $\forall \; u,v \in B_\mu(0) = \{w \in V\colon |||w||| < \mu,\ \mu > 0\}$,

$$[A(u,u)-A(v,u),u-v] \geq -H\left(\mu, \|u-v\|_{L^p(0,T;X)}\right) \tag{6.4}$$

where $H\colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ $(\mathbb{R}^+ = [0,\infty))$ is a function continuous in each of its arguments with the property that

$$\lim_{\theta \downarrow 0} \frac{1}{\theta} H(x,\theta y) = 0 \quad \forall \quad x,y \in \mathbb{R}^+ \tag{6.5}$$

and $L^p(0,T;X)$ is as in Theorem 6.1.

BIII. If $u_n \longrightarrow u$ weakly in $\mathcal{V}$ of (6.1), then

$$\left.\begin{aligned} &\liminf_{n\to\infty} [A(v,u_n) - A(v,u),u_n - u] \geq 0 \quad \forall \quad v \in V \\ \text{and} \quad &\liminf_{n\to\infty} [A(v,u_n) - A(v,u),w] = 0 \quad \forall \quad v,w \in V \end{aligned}\right\} \tag{6.6}$$

BIV. $A\colon V \to V'$ is bounded.

The importance of these conditions is made clear in the following theorem proved in [13].

<u>Theorem 6.2.</u> Let $A\colon V \to V'$ satisfy conditions (BI), (BII), (BIII), and (BIV). Then $A\colon \mathcal{V} \to \mathcal{V}'$ is $\mathcal{V}$-pseudomonotone. $\square$

PART II. A MODEL PSEUDOMONOTONE DIFFUSION PROBLEM

7. Existence Analysis

In this part, we are concerned with the qualitative analysis, Galerkin and Faedo-Galerkin approximations of the following generalized nonlinear diffusion problem: Let Ω be an arbitrary bounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$, and $0 < T < \infty$. Given data f in $\Omega \times (0,T)$ and initial data u_o on Ω, find $u = u(x,t)$, $(x,t) \in \Omega \times (0,T)$, such that

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nabla \cdot \underset{\sim}{a}(\nabla u) + b(u,\nabla u) &= f \,, \quad \text{in} \quad Q = \Omega \times (0,T) \\ u &= 0 \,, \quad \text{on} \quad \Sigma = \partial\Omega \times (0,T) \\ u(\cdot,0) &= u_o, \quad \text{on} \quad \Omega \end{aligned}\right\} \tag{7.1}$$

where

$$\left.\begin{aligned} &\underset{\sim}{a}(\nabla u) = a\nabla u + k|\nabla u|^{p-2}\,\nabla u \\ &a,k \in L^\infty(\Omega) \,, \quad 2 \le p < \infty \\ &a(x) \ge a_o \ge 0 \quad \text{and} \quad k(x) \ge k_o > 0 \quad \text{a.e.} \quad x \in \Omega \end{aligned}\right\} \tag{7.2}$$

and $b(u,\nabla u) = b(x,u(x,t), \nabla u(x,t))$ is subject at a.e. $(x,t) \in Q$ to the conditions:

CI. $|b(\zeta,\underset{\sim}{\zeta})| \le c|\zeta|^q\,|\underset{\sim}{\zeta}|^r \,, \quad \forall \quad (\zeta,\underset{\sim}{\zeta}) \in \mathbb{R} \times \mathbb{R}^n$

$$\left.\begin{aligned} &q = 0 \quad \text{or} \quad q \ge 1 \,, \quad r = 0 \quad \text{or} \quad r \ge 1 \\ &1 \le q + r < p - 1 \end{aligned}\right\} \tag{7.3}$$

CII. $b(\zeta,\underset{\sim}{\zeta})$ is totally Fréchet differentiable in $\mathbb{R} \times \mathbb{R}^n$ and its partial derivatives $\partial_\zeta b\colon \mathbb{R} \times \mathbb{R}^n \to L(\mathbb{R},\mathbb{R})$ and $\partial_{\underset{\sim}{\zeta}} b\colon \mathbb{R} \times \mathbb{R}^n \to L(\mathbb{R}^n,\mathbb{R})$ are such that, for (q,r) satisfying (7.3) and $\forall \quad (\zeta,\underset{\sim}{\zeta}) \in \mathbb{R} \times \mathbb{R}^n$,

$$|\partial_\zeta b(\zeta,\underset{\sim}{\zeta})| \le c_q\,|\zeta|^{q-1}\,|\underset{\sim}{\zeta}|^r \,, \quad \text{if} \quad q \ne 0$$

$$|\partial_{\underset{\sim}{\zeta}} b(\zeta,\underset{\sim}{\zeta})| \le c_r\,|\zeta|^q\,|\underset{\sim}{\zeta}|^{r-1} \,, \quad \text{if} \quad r \ne 0$$

The case $r = 0$ will be understood as $b = b(u)$ (not function of ∇u), and the case $q = 0$ as $b = b(\nabla u)$ (not function of u).

In this case, we take as spaces V and H the usual Sobolev spaces

$$\left.\begin{aligned} V &= W_o^{1,p}(\Omega)\ , \quad 2 \le p < \infty \\ H &= L^2(\Omega) \end{aligned}\right\} \tag{7.4}$$

Then, with the conventions of Section 2 in force, the model problem (7.1) assumes the following form:

Find $u \in \mathcal{W}$ such that

$$\left.\begin{aligned} \frac{\partial u}{\partial t} + A(u) &= f \quad , \quad f \text{ given in } \mathcal{V}' \\ u(0) &= u_o \ , \quad u_o \text{ given in } L^2(\Omega) \end{aligned}\right\} \tag{7.5}$$

where $A\colon \mathcal{V} \to \mathcal{V}'$ is defined by

$$\left.\begin{aligned} [A(u),v] &= [A_1(u),v] + [A_2(u),v] \\ [A_1(u),v] &= \int_Q \underset{\sim}{a}\big(x,\nabla u(x,t)\big)\cdot\nabla v(x,t)\ dxdt \\ [A_2(u),v] &= \int_Q b\big(x,u(x,t),\nabla u(x,t)\big)v(x,t)\ dxdt \end{aligned}\right\} \tag{7.6}$$

in which $\underset{\sim}{a}(\nabla u)$ is as defined in (7.2) and $b(u,\nabla v)$ is subject to conditions (CI) and (CII).

We now proceed to establish the existence of solutions to problem (7.5). The following two theorems determine fundamental properties of the operator A.

<u>Theorem 7.1.</u> Let $A\colon \mathcal{V} \to \mathcal{V}'$ be the operator defined in (7.6). Then i) A is bounded, ii) A is coercive, and iii) A is locally Lipschitz continuous in the sense that $\forall\ u,v \in B_\mu(0) = \{v \in \mathcal{V}\colon |||v||| < \mu\ ,\ \mu > 0\}$, $w \in \mathcal{V}$, there is a positive constant $C(\mu)$ such that

$$|[A(u)-A(v),w]| \le C(\mu)\,|||u-v|||\ \ |||w||| \tag{7.7}$$

<u>Proof.</u> We shall use the notation $a_\infty = \|a\|_{L^\infty(\Omega)}$, $k_\infty = \|k\|_{L^\infty(\Omega)}$, and $\|\cdot\|_{s,Q} = \|\cdot\|_{L^s(Q)}$

i) Applying Hölder's inequality, we obtain, $\forall\ v,w \in \mathcal{V}$

$$\begin{aligned} |[A_1(v),w]| &\le \int_Q \left\{ a_\infty|\nabla v| + k_\infty|\nabla v|^{p-1}\right\}|\nabla w|\,dQ \\ &\le \left\{ a_\infty \text{mes}(Q)^{(p-2)/p}\|\nabla v\|_{p,Q} + k_\infty\|\nabla v\|^{p-1}_{p,Q}\right\}\|\nabla w\|_{p,Q} \end{aligned}$$

$$|[A_2(v),w]| \leq \int_Q c|v|^q|\nabla v|^r|w|dQ$$
$$\leq c \text{ mes } (Q)^{(p-1-q-r)/p} \|v\|^q_{p,Q} \|\nabla v\|^r_{p,Q} \|w\|_{p,Q}$$

Hence,

$$|||A(v)|||_* \leq a_\infty \text{ mes}(Q)^{\frac{p-2}{p}} |||v||| + k_\infty |||v|||^{p-1}$$
$$+ c \text{ mes}(Q)^{\frac{p-1-q-r}{p}} |||v|||^{q+r}, \quad \forall \quad v \in V \tag{7.8}$$

ii) From Friedrichs' inequality, it follows that, $\forall \quad v \in L^s(0,T;W_o^{1,s}(\Omega))$, $1 \leq s < \infty$,

$$\|v\|^s_{L^s(Q,T;W_o^{1,s}(\Omega))} = \|v\|^s_{s,Q} + \|\nabla v\|^s_{s,Q}$$
$$\leq (c^s(s,n) \text{ mes}(\Omega)^{s/n} + 1)\|\nabla v\|^s_{s,Q} \tag{7.9}$$

Thus,

$$[A(v),v] \geq \int_Q (a_o|\nabla v|^2 + k_o|\nabla v|^p)dQ - \int_Q c|v|^{q+1}|\nabla v|^r dQ$$
$$\geq a_o\|\nabla v\|^2_{2,Q} + k_o(1 + c^p(p,n) \text{ mes}(\Omega)^{p/n})^{-1} |||v|||^p$$
$$- c \text{ mes}(Q)^{\frac{p-1-q-r}{p}} |||v|||^{1+q+r}, \quad \forall \quad v \in V \tag{7.10}$$

But using Young's inequality (2.12) in the last term in (7.10) and choosing b small enough leads to

$$[A(v),v] \geq a_o\|\nabla v\|^2_{2,Q} + \gamma_1|||v|||^p - \gamma_2 T, \quad \forall \quad v \in V \tag{7.11}$$

where γ_1 and γ_2 are > 0. Therefore,

$$\frac{[A(v),v]}{|||v|||} \geq \gamma \; |||v|||^{p-1} - \frac{\gamma_2 T}{|||v|||} \to +\infty \text{ as } |||v||| \to \infty$$

iii) By the inequality in $\mathbb{R}^n$ [15]

$$\left.\begin{aligned} &||x|^{r-2}x - |y|^{r-2}y| \leq c\{|x| + |y|\}^{r-2}|x-y| \\ &c = \sqrt{r-1} \text{ if } 2 \leq r \leq 3, \quad c = r-1 \text{ if } 3 \leq r < \infty \end{aligned}\right\} \tag{7.12}$$

and Holder's inequality, we obtain, $\forall \quad u,v \in B_\mu(0) \subset V$, $w \in V$,

$$|[A_1(u)-A_1(v),w]|$$
$$\leq \int_Q \left\{a_\infty|\nabla(u-v)| + k_\infty c(p)(|\nabla u| + |\nabla v|)^{p-2}|\nabla(u-v)|\right\}|\nabla w|dQ$$
$$\leq \left\{a_\infty \text{ mes}(Q)^{\frac{p-2}{p}} + k_\infty c(p)(2\mu)^{p-2}\right\} |||u-v||| \; |||w||| \tag{7.13}$$

We now use hypothesis (CII). First we observe that

$$[A_2(u)-A_2(v),w] = \int_Q\int_0^1 \frac{db(\xi,\nabla\xi)}{d\theta}\, w\, d\theta dQ$$

$$\int_Q\int_0^1 \{\partial_\xi b(\xi,\nabla\xi)\eta + \partial_{\nabla\xi} b(\xi,\nabla\xi)\cdot\nabla\eta\} w\, d\theta dQ \qquad (7.14)$$

where $\xi = v + \theta\eta$, $\eta = u-v$ and $\theta \in [0,1]$. Hence, because of (CII) and Hölder's inequality,

$$\begin{aligned} &|[A_2(u) - A_2(v),w]| \\ &\le \int_0^1\int_Q \left\{ c_q|\xi|^{q-1}|\nabla\xi|^r|\eta| + c_r|\xi|^q|\nabla\xi|^{r-1}|\nabla\eta| \right\} |w|\, dQ d\theta \\ &\le (c_q + c_r)\ \mathrm{mes}\ (Q)^{\frac{p-1-q-r}{p}} \int_0^1 |||\xi|||^{q+r-1} d\theta\, |||\eta|||\ \ ||w||_{p,Q} \end{aligned}$$

Then, since $u,v \in B_\mu(0) \subset V$,

$$\left.\begin{aligned} &|[A_2(u)-A_2(v),w]| \le \gamma_3\mu^{q+r-1}|||u-v|||\ \ ||w||_{L^p(Q)} \\ &\gamma_3 = (c_q + c_r)\ \mathrm{mes}(Q)^{\frac{p-1-q-r}{p}} \end{aligned}\right\} \qquad (7.15)$$

Therefore, from estimates (7.13) and (7.15), (7.7) follows and this completes the proof of the theorem. □

The next property of A , established below, is crucial, not only in proving the existence of solutions to (7.5) but in subsequent studies of approximations.

<u>Theorem 7.2.</u> The operator $A: V \to V'$ defined in (7.6) satisfies the following nonlinear Gårding-type inequality:

$$\begin{aligned} [A(u)-A(v),u-v] \ge \alpha_o a_o ||u-v||^2_{L^2(0,T,H_o^1(\Omega))} + \alpha_1 |||u-v|||^p \\ - \alpha_2(\mu)||u-v||^{p'}_{L^p(Q)} \end{aligned} \qquad (7.16)$$

$\forall\ u,v \in B_\mu(0) = \{w \in V:\ |||w||| < \mu\ ,\ \mu > 0\}$

Here $H_o^1(\Omega) = W_o^{1,2}(\Omega)$ and α_o , α_1 , $\alpha_2(\mu)$ are constants satisfying

$\alpha_o > 0$, $\alpha_1 > 0$, $\alpha_2(\mu) > 0$.

<u>Proof.</u> We observe that, $\forall\ u,v \in V$,

$$[A(u) - A(v), u - v] \geq [A_1(u) - A_1(v), u - v] - |[A_2(u) - A_2(v), u - v]| \tag{7.17}$$

From the inequality in $\mathbb{R}^n$ [15]

$$(|x|^{r-2}x - |y|^{r-2}y, x-y) \geq 2^{1-r}|x-y|^r \ , \quad 2 \leq r < \infty \tag{7.18}$$

and (13.9), it follows that

$$\begin{aligned}[A_1(u)-A_1(v),u-v] &\geq \int_Q \left\{ a_o|\nabla(u-v)|^2 + k_o 2^{1-p}|\nabla(u-v)|^p \right\} dQ \\ &\geq a_o\left(1+c^2(2,n)\ \mathrm{mes}(\Omega)^{2/n}\right)^{-1} |||u-v|||^2_{L^2(0,T;W_o^{1,2}(\Omega))} \\ &\quad + k_o 2^{1-p}\left(1+c^p(p,n)\ \mathrm{mes}(\Omega)^{p/n}\right)^{-1} |||u-v|||^p \end{aligned} \tag{7.19}$$

On the other hand, according to (7.15) and Young's inequality (2.12), for $u,v \in B_\mu(0) \subset V$,

$$|[A_2(u)-A_2(v),u-v]| \leq \frac{b^p}{p} |||u-v|||^p + \frac{\gamma_3^{p'} \mu^{(q+r-1)p'}}{p' b^{p'}} \| u-v\|^{p'}_{L^p(Q)} \tag{7.20}$$

Therefore, introducing (7.19) and (7.20) into (7.18) and choosing b small enough, the desired result (7.16) is obtained. □

<u>Theorem 7.3.</u> For any data $f \in V'$ and $u_o \in L^2(\Omega)$, there exists at least one solution $u \in W$ to problem (7.5).

<u>Proof.</u> Theorem 7.1 confirms that conditions (BI), (BIII) with $\mathcal{Y} = W$, and (BIV) of Theorem 6.2 are satisfied ((BIII) is trivially satisfied). Condition (BII) with $X = L^p(\Omega)$ ($\hookrightarrow V'$ and in which V is compact (cf., e.g., [1])) also holds since, by virtue of (7.16),

$$[A(u)-A(v),u-v] \geq -H(\mu, \| u-v\|_{L^p(Q)})$$

$\forall$ $u,v \in B_\mu(0) \subset V$, where

$$H(\mu,y) = \alpha_2(\mu)y^{p'} \ , \quad y \in \mathbb{R}^+ \ , \quad p' = p/p(p-1)$$

Therefore, A is coercive and W-pseudomonotone from $V \to V'$ and, by virtue of

Theorem 3.1, the assertion of the theorem follows. □

Remark 7.1. From the proofs of Theorems 7.1 and 7.2, it is apparent that the operator A of (7.6) regarded as a map from V into V', is bounded, coercive and locally Lipschitz continuous, and satisfies the Gårding-type inequality

$$\langle A(u)-A(v),u-v\rangle \geq \alpha_o a_o \| u-v\|^2_{H^1_o(\Omega)} + \alpha_1 \| u-v\|^p - \hat{\alpha}_2(\rho)\| u-v\|^{p'}_{L^p(\Omega)} \;,\quad \forall\; u,v \in B_\rho(0) \subset V \tag{7.21}$$

According to Oden [13], $A: V \to V'$ is necessarily (V-)pseudomonotone. Hence, from the theory of pseudomonotone elliptic equations (cf. [11]), A is surjective from $V \to V'$; i.e., there exists at least one solution in V to the stationary problem

$$A(u) = f \;,\quad f \text{ given in } V' \tag{7.22}$$

The evolution problem (7.5) possesses at least one equilibrium state for each $f \in V'$. □

8. Sufficient Conditions for Uniqueness

We now proceed to determine sufficient conditions for uniqueness, of solutions to the pseudomonotone diffusion problem (715).

In the case of monotone parabolic problems, "monotonicity" $\Longrightarrow$ "uniqueness" and this follows from the differential inequality of Carathéodory type: $d|u(t)-v(t)|^2/dt \leq 0$, a.e. $t \in [0,T]$, $|u(0)-v(0)|^2 = 0$, whose unique solution is $|u(t)-v(t)|^2 = 0$; u and v are supposed solutions of the problem. This suggests that in the non-monotone case with Gårding-type inequality, the possibility of establishing a differential inequality of the form

$$\left.\begin{aligned} &\frac{d}{dt}|u(t)-v(t)|^s \leq \alpha|u(t)-v(t)|^s \\ &\alpha \in R\,,\quad 2 \leq s < \infty\,,\quad \text{for a.e. } t \in [0,T] \\ &|u(0)-v(0)|^s = 0 \end{aligned}\right\} \tag{8.1}$$

would be sufficient for concluding uniqueness. Indeed, from Theorem 3 of Olech and Opial [14], $|u(t)-v(t)|^s = 0$ is the unique solution to (8.1). We show that in certain particular cases and, in general, for sufficiently smooth solutions of problem (7.5), this is the case.

Theorem 8.1. Let $u \in \mathcal{W}$ be a solution of problem (7.5). Then u is unique in the following three cases:

$$\left.\begin{array}{ll} \text{i)} & r = 0 \text{ and } q = 1 \\ \text{ii)} & r = 0 \text{ and } n < p \\ \text{iii)} & a_o > 0 \text{ and } u \in L^{\infty}(0,T;W_o^{1,\infty}(\Omega))^* \end{array}\right\} \tag{8.2}$$

Proof. Assume that $u = u(t;f,u_o)$ and $v = v(t;f,u_o)$ are two solutions of problem (7.5) and define $\eta = u-v$. As is apparent from (7.19),

$$\langle A_1(u(t))-A_1(v(t)),\eta(t)\rangle \geq \hat{\alpha}_o a_o \| \eta(t) \|^2_{H_o^1(\Omega)} + \hat{\alpha}_1 \| \eta(t) \|^p$$

$$\text{for a.e. } t \in [0,T] \tag{8.3}$$

where $\hat{\alpha}_o > 0$ and $\hat{\alpha}_1 > 0$. Thus, from the difference of the equations satisfied by u and v, we obtain the integral inequality

$$\frac{1}{2} |\eta(\tau)|^2 + \hat{\alpha}_o a_o \int_0^\tau \| \eta(\tau) \|^2_{H_o^1(\Omega)} dt$$

$$\leq \left| \int_0^\tau \langle A_2(u(t))-A_2(v(t)),\eta(t)\rangle\, dt \right|$$

$$\hat{\alpha}_o > 0, \quad \forall \ \tau \in [0,T] \tag{8.4}$$

We now estimate the right-hand side term via the formula (7.14) with $w = \eta$.

i) $r = 0$ and $q = 1$. In this case we have the estimate

$$\left| \int_0^\tau \langle A_2(u(t))-A_2(v(t)),\eta(t)\rangle\, dt \right| \leq c_q \int_0^\tau |\eta(t)|^2 dt\,,$$

$$\forall \ \tau \in [0,T] \tag{8.5}$$

which combined with (8.4) gives the integral inequality

*In this case, the question of existence appears to be open.

$$|\eta(\tau)|^2 \le 2c_q\int_0^\tau |\eta(t)|^2 dt\ , \quad \forall \quad \tau \in [0,T] \tag{8.6}$$

But this is equivalent to (8.1) with $\alpha = 2c_q$ and $s = 2$. Consequently, $\eta = 0$.

ii) $r = 0$ and $n < p$. From the Sobolev embedding theorem (cf. [1, Chap. 5]), $W_o^{1,p}(\Omega)$ is continuously embedded in $C_B(\Omega) = \{v \in C(\Omega):\ v \text{ bounded in } \Omega\}$ whenever $n < p$. Then

$$V = L^p(0,T;W_o^{1,p}(\Omega)) \hookrightarrow L^p(0,T;L^\infty(\Omega))\ , \quad n < p \tag{8.7}$$

Let μ be chosen such that $u,v \in B_\mu(0) \subset V$. Then, from (7.14) with $w = \eta$ and using (8.7), we obtain, $\forall\ \tau \in [0,T]$,

$$\begin{aligned}
&\left|\int_0^\tau \langle A_2(u(t))-A_2(v(t)),\eta(t)\rangle\, dt\right| \\
&\quad\le \int_0^1\int_0^\tau\int_\Omega c_q|\xi(x,t)|^{q-1}|\eta(x,t)|^2 dxdtd\theta \\
&\quad\le \int_0^1\int_0^\tau c_q\|\xi(t)\|^{q-1}_{L^\infty(\Omega)}|\eta(t)|^2 dtd\theta \\
&\quad\le c_q\mu^{q-1}\left(\int_0^\tau |\eta(t)|^s dt\right)^{\frac{2}{s}}\ , \quad 2 \le s = \frac{2p}{p+1-q} < p
\end{aligned} \tag{8.8}$$

Introducing this estimate into (8.4) produces the integral inequality

$$|\eta(\tau)|^s \le \left(2c_q\mu^{q-1}\right)^{s/2}\int_0^\tau |\eta(t)|^s dt\ , \quad \forall \quad \tau \in [0,T] \tag{8.9}$$

which is equivalent to the differential inequality (8.1) with $\alpha = \left(2c_q\mu^{q-1}\right)^{s/2}$ and $2 \le s = sp/(p+1-q) < p$. Therefore, $\eta = 0$.

iii) $a_o > 0$ and $u \in L^\infty(0,T;W_o^{1,\infty}(\Omega))$. Let $\mu > 0$ be such that $u,v \in B_\mu(0) \subset L^\infty(0,T;W_o^{1,\infty}(\Omega))$. Then, from (7.14) with $w = \eta$, we obtain $\forall\ \tau \in [0,T]$.

$$\begin{aligned}
&\left|\int_0^\tau \langle A_2(u(t))-A_2(v(t)),\eta(t)\rangle\, dt\right| \\
&\quad\le \int_0^1\int_0^\tau\int_\Omega \Big\{c_q|\xi(x,t)|^{q-1}|\nabla\xi(x,t)|^r|\eta(x,t)| \\
&\qquad\qquad + c_r|\xi(x,t)|^q|\nabla\xi(x,t)|^{r-1}|\nabla\eta(x,t)|\Big\}|\eta(x,t)|dxdtd\theta \\
&\quad\le \mu^{q+r-1}\left(c_q c(2,n)\ \text{mes}(\Omega)^{1/n}+c_r\right)\int_0^\tau \|\eta(t)\|_{H_o^1(\Omega)}|\eta(t)|dt
\end{aligned} \tag{8.10}$$

Hence, since by hypothesis $a_o > 0$, we can apply Young's inequality with constant, e.g., $b = \sqrt{\alpha_o a_o}$, to obtain the upper bound for (8.10)

$$\frac{\alpha_o a_o}{2}\int_0^\tau \| \eta(t) \|^2_{H^1_o(\Omega)} dt + \frac{\alpha}{2}\int_0^\tau |\eta(t)|^2 dt$$

where $\alpha = \alpha(1/b^2) > 0$. Combining these results with (8.4) gives

$$|\eta(\tau)|^2 \leq \alpha \int_0^\tau |\eta(t)|^2 dt \tag{8.11}$$

and, consequently, (8.1) holds with $s = 2$ and $\eta = 0$. □

9. Galerkin Approximations

In this section, we study Galerkin approximations of the model problem (7.5) which are based on an elliptic regularization of (7.5) obtained using the ideas described in Section 4. We will establish some results on the strong convergence of such approximations.

For the model problem (7.5), we introduce the corresponding elliptic regularization (4.2):

$$\begin{aligned} \varepsilon(\dot{u}_\varepsilon,\dot{v})_H - (u_\varepsilon,\dot{v})_H + (u_\varepsilon(T),v(T)) + [A(u_\varepsilon),v] \\ = [f,v] + (u_o,v(0)) , \quad \forall \quad v \in U \end{aligned} \tag{9.1}$$

where A is the operator defined in (7.6). According to Theorems 7.1 and 4.1, a solution $u_\varepsilon \in U$ exists to such a regularization for every $\varepsilon > 0$. Moreover, according to Section 5, in the sense of $V' \subset L(\mathcal{D}((0,T)),W^{-1,p'}(\Omega))$, u_ε satisfies the distributional equation

$$\left.\begin{aligned} &-\varepsilon\ddot{u}_\varepsilon + \dot{u}_\varepsilon + A(u_\varepsilon) = f \quad \text{in} \quad V' \\ &-\varepsilon\dot{u}_\varepsilon(0) + u_\varepsilon(0) = u_o \quad \text{in} \quad L^2(\Omega) \\ &\dot{u}_\varepsilon(T) = 0 \quad \text{in} \quad L^2(\Omega) \end{aligned}\right\} \tag{9.2}$$

which is equivalent to (5.1), and for any sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset U$ of solutions, there exists a subsequence, also denoted $\{u_\varepsilon\}_{\varepsilon>0}$ such that, as $\varepsilon \to 0^+$, u_ε converges weakly to a solution u of (7.5) in the sense of Lemma 5.2 with $\chi = A(u)$.

To construct Galerkin approximations of (9.1), we introduce a family of subspaces $\{\mathcal{U}_h\}_{0<h\leq 1}$ of $\mathcal{U}$ such that i) $\mathcal{U}_h$ is finite-dimensional with basis functions $\{\phi_1, \phi_2, \ldots, \phi_{m_h}\}$, with dimension $m_h \to \infty$ as $h \to 0^+$ and ii) $\bigcup_h \mathcal{U}_h$ is dense in $\mathcal{U}$. A Galerkin approximation of (9.1) involves seeking a function $U^h_\varepsilon \in \mathcal{U}_h$ such that

$$\varepsilon(\dot{U}^h_\varepsilon, \dot{\phi}_k)_H - (U^h_\varepsilon, \dot{\phi}_k)_H + (U^h_\varepsilon(T), \phi_k(T)) + [A(U^h_\varepsilon), \phi_k]$$
$$= [f, \phi_k] + (u_o, \phi_k(0)) \;, \quad k = 1,2,\ldots,m_h \tag{9.3}$$

The solvability in $\mathcal{U}_h$ of (9.3) is assured by Lemma 4.1. Similarly as in the proof of Lemma 5.1, if $\{U^h_\varepsilon\}_{0<h\leq 1}$ is a sequence of Galerkin approximate solutions, it can be shown that there exist constants K_1, K_2, K_3 and K_4, independent of h, such that $|||U^h_\varepsilon||| \leq K_1$, $|\dot{U}^h_\varepsilon|_H \leq K_2$, $|U^h_\varepsilon(0)| \leq K_3$ and $|U^h_\varepsilon(T)| \leq K_4$. Then, via weak compactness and pseudomonotonicity arguments (as those used in Section 5), it follows that there exists a function u_ε and a subsequence, also denoted $\{U^h_\varepsilon\}_{0<h\leq 1}$, such that, as $h \to 0^+$,

$$\left.\begin{array}{lll} U^h_\varepsilon \rightharpoonup u_\varepsilon & \text{weakly in} & V \\ \dot{U}^h_\varepsilon \rightharpoonup \dot{u}_\varepsilon & \text{weakly in} & L^2(Q) \\ A(U^h_\varepsilon) \rightharpoonup A(u_\varepsilon) & \text{weakly in} & V' \\ U^h_\varepsilon(0) \rightharpoonup u_\varepsilon(0) & \text{weakly in} & L^2(\Omega) \\ U^h_\varepsilon(T) \rightharpoonup u_\varepsilon(T) & \text{weakly in} & L^2(\Omega) \end{array}\right\} \tag{9.4}$$

We will now demonstrate that for our model problem (7.5) much stronger results can be obtained.

<u>Theorem 9.1.</u> Let $\{u_\varepsilon\}_{\varepsilon>0} \subset \mathcal{U}$ be a weakly convergent subsequence of solutions to problem (9.1) and let $u \in \mathcal{W}$ be the corresponding weak limit, solution of problem (7.5). Then, as $\varepsilon \to 0^+$,

$$\left.\begin{array}{lll} u_\varepsilon \to u & \text{strongly in} & V \\ \sqrt{\varepsilon}\, \dot{u}_\varepsilon \to 0 & \text{strongly in} & L^2(Q) \\ u_\varepsilon(0) \to u_o & \text{strongly in} & L^2(\Omega) \\ u_\varepsilon(T) \to u(T) & \text{strongly in} & L^2(\Omega) \end{array}\right\} \tag{9.5}$$

Proof. We regard equation (7.5) as holding on $\mathcal{U}$ and substract (9.1) from it. The following orthogonality condition is obtained:

$$-(\varepsilon\dot{u}_\varepsilon,\dot{v})_H + (u_o-u_\varepsilon(0),v(0)) + [\dot{u}-\dot{u}_\varepsilon,v] + [A(u)-A(u_\varepsilon),v] = 0 \quad \forall \quad v \in \mathcal{U} \tag{9.6}$$

According to the a priori bound $(5.1)_1$, there is a $\mu > 0$ independent of ε, such that u_ε, $u \in B_\mu(0) \subset \mathcal{V}$. Hence, using formula (2.9) and the Gårding-type inequality of (7.16), we see that

$$\begin{aligned}|u_o-u_\varepsilon(0)|^2 &+ [\dot{u}-\dot{u}_\varepsilon,u-u_\varepsilon] + [A(u)-A(u_\varepsilon),u-u_\varepsilon] \\ &\geq \frac{1}{2}\,|u_o-u_\varepsilon(0)|^2 + \frac{1}{2}\,|u(T)-u_\varepsilon(T)|^2 \\ &\quad + \alpha_1 |||u-u_\varepsilon|||^p - \alpha_2(\mu)\,\| u-u_\varepsilon\|^{p'}_{L^p(Q)}\end{aligned} \tag{9.7}$$

Next, combining these two results, we conclude that

$$\begin{aligned}\frac{1}{2}\,|u_o-u_\varepsilon(0)|^2 &+ \frac{1}{2}\,|u(T)-u_\varepsilon(T)|^2 + \alpha_1 |||u-u_\varepsilon|||^p \\ &\leq \alpha_2(\mu)\,\| u-u_\varepsilon\|^{p'}_{L^p(Q)} + (u_o-u_\varepsilon(0),u_o-v(0)) \\ &\quad + [\dot{u}-\dot{u}_\varepsilon,u-v] + [A(u)-A(u_\varepsilon),u-v] + (\varepsilon\dot{u}_\varepsilon,\dot{v})_H - |\sqrt{\varepsilon}\,\dot{u}_\varepsilon|^2_H \\ &\qquad \forall \quad v \in \mathcal{U}\end{aligned} \tag{9.8}$$

Due to the compact embedding of $\mathcal{W}$ in $L^p(Q)$ (cf. Theorem 6.1) and the weak convergence result of Section 5, (9.5) follows. □

Theorem 9.2. Let $\{U^h_\varepsilon \in \mathcal{U}_h\}_{0<h\leq 1}$ be a subsequence of Galerkin approximate solutions defined by (9.3), converging weakly, in the sense of (9.4), to a solution $u_\varepsilon \in \mathcal{U}$ of problem (9.1). Then, for fixed $\varepsilon > 0$, as $h \to 0^+$

$$\left.\begin{aligned} U^h_\varepsilon &\to u_\varepsilon && \text{strongly in } \mathcal{V} \\ \dot{U}^h_\varepsilon &\to \dot{u}_\varepsilon && \text{strongly in } L^2(Q) \\ U^h_\varepsilon(0) &\to u_\varepsilon(0) && \text{strongly in } L^2(\Omega) \\ U^h_\varepsilon(T) &\to u_\varepsilon(T) && \text{strongly in } L^2(\Omega)\end{aligned}\right\} \tag{9.9}$$

Proof. We follow similar arguments to those given previously. Restricting (9.1) to $\mathcal{U}_h$ and subtracting (9.3) from it, we obtain the orthogonality condition

$$\varepsilon(\dot{u}_\varepsilon-\dot{U}^h_\varepsilon,\dot{W})_H - (u_\varepsilon-U^h_\varepsilon,\dot{W})_H + (u_\varepsilon(T)-U^h_\varepsilon(T),W(T))$$
$$+ [A(u_\varepsilon)-A(U^h_\varepsilon),W] = 0 \ , \quad \forall \quad W \in \mathcal{U}_h \tag{9.10}$$

Now, from $(5.1)_1$ and since $|||U^h_\varepsilon||| \leq K_1 \quad \forall \quad h \in (0,1]$ there is a $\mu > 0$, independent of h such that $U^h_\varepsilon, u_\varepsilon \in B_\mu(0) \subset V$. Then, by virture of (2.9) and (7.16), it follows that

$$\varepsilon|\dot{u}_\varepsilon-\dot{U}^h_\varepsilon|^2_H - (u_\varepsilon-U^h_\varepsilon,\dot{u}_\varepsilon-\dot{U}^h_\varepsilon)_H + |u_\varepsilon(T)-U^h_\varepsilon(T)|^2$$
$$+ [A(u_\varepsilon)-A(U^h_\varepsilon),u_\varepsilon-U^h_\varepsilon]$$
$$\geq \varepsilon|\dot{u}_\varepsilon-\dot{U}^h_\varepsilon|^2_H + \frac{1}{2}\ |u_\varepsilon(0)-U^h_\varepsilon(0)|^2 + \frac{1}{2}\ |u_\varepsilon(T)-U^h_\varepsilon(T)|^2$$
$$+ \alpha_o\| u_\varepsilon-U^h_\varepsilon\|^2_{L^2(0,T;H^1_o(\Omega))} + \alpha_1|||u_\varepsilon-U^h_\varepsilon|||^p$$
$$- \alpha_2(\mu)\| u_\varepsilon-U^h_\varepsilon\|^{p'}_{L^p(Q)} \tag{9.11}$$

Therefore, combining (9.10) and (9.11)

$$\varepsilon|\dot{u}_\varepsilon-\dot{U}^h_\varepsilon|^2_H + \frac{1}{2}\ |u_\varepsilon(0)-U^h_\varepsilon(0)|^2 + \frac{1}{2}\ |u_\varepsilon(T)-U^h_\varepsilon(T)|^2$$
$$+ \alpha_o\| u_\varepsilon-U^h_\varepsilon\|^2_{L^2(0,T;H^1_o(\Omega))} + \alpha_1|||u_\varepsilon-U^h_\varepsilon|||^p$$
$$\leq \alpha_2(\mu)\| u_\varepsilon-U^h_\varepsilon\|^{p'}_{L^p(Q)} + \varepsilon(\dot{u}_\varepsilon-\dot{U}^h_\varepsilon,\dot{u}_\varepsilon-\dot{W})_H$$
$$- (u_\varepsilon-U^h_\varepsilon,\dot{u}_\varepsilon-\dot{W})_H + (u_\varepsilon(T)-U^h_\varepsilon(T),u_\varepsilon(T)-W(T))$$
$$+ [A(u_\varepsilon)-A(U^h_\varepsilon),u_\varepsilon-W] \quad \forall \quad W \in \mathcal{U}_h \tag{9.12}$$

But, according to Theorem 6.1, $\mathcal{U}$ is compactly embedded in $L^p(Q)$ and U^h_ε converges weakly to u_ε in the sense of (9.4). Hence, the right side of (9.12) $\to 0$ as $h \to 0^+$ and this proves the theorem. □

We next give an error estimate for the Galerkin approximations of the regularized elliptic problem (9.1).

<u>Theorem 9.3.</u> For fixed $\varepsilon > 0$, let $u_\varepsilon \in \mathcal{U}$ be a solution of problem (9.1) which is the strong limit (in the sense of (9.9)) of a subsequence of Galerkin approximate solutions $\{U^h_\varepsilon \in \mathcal{U}_h\}_{0<h\leq 1}$ defined by (9.3). Then the following

approximation error estimate holds $\forall\ W \in \mathcal{U}_h$:

$$\frac{1}{2} C_1 |u_\varepsilon(0)-U_\varepsilon^h(0)|^2 + \frac{1}{2} |u_\varepsilon(T)-U_\varepsilon^h(T)|^2$$

$$+ \alpha_o a_o \| u_\varepsilon - U_\varepsilon^h \|^2_{L^2(0,T;H_o^1(\Omega))} + \tilde{\alpha}\ |||u_\varepsilon - U_\varepsilon^h|||^p + \tilde{\varepsilon} |\dot{u}_\varepsilon - \dot{U}_\varepsilon^h|^2_H$$

$$\leq \alpha_2 \| u_\varepsilon - U_\varepsilon^h \|^{p'}_{L^p(Q)} + C_2 |u_\varepsilon(0)-W(0)|^2 + C_3 |u_\varepsilon - W|^2_H$$

$$+ \tilde{C} |||u_\varepsilon - W|||^{p'} + C_4 |\dot{u}_\varepsilon - \dot{W}|^2_H \tag{9.13}$$

where C_i, $i = 1,\ldots,4,\alpha_o$, $\tilde{\alpha}_1 = \tilde{\alpha}_1(\alpha_1)$, $\alpha_2 = \alpha_2(T,\mu)$ $\tilde{\varepsilon} = \tilde{\varepsilon}(\varepsilon)$ and $\tilde{C} = \tilde{C}(C(T,\mu))$ are strictly positive constants. Here $C(T,\mu)$ is the local Lipschitz continuity constant of (7.7).

Proof. The estimate (9.13) follows directly from (9.12) upon applying formula (2.9), the local Lipschitz continuity of A , (7.7), and inequality (2.12). □

10. Faedo-Galerkin Approximations

We are concerned here with Faedo-Galerkin approximations of the model pseudomonotone diffusion problem (7.5). We note that this type of approximation process is not necessarily well-defined for non-monotone parabolic problems: the corresponding weak convergence is a conditional property. We shall show that Faedo-Galerkin approximate solutions to problem (7.5) exist and are unique, and determine sufficient conditions for weak and strong convergence.

Let $\{V_h\}_{0<h\leq 1}$ be a family of finite-dimensional subspaces approximating the space $V(= W_o^{1,p}(\Omega))$ in the following sense: (i) $\{\psi_1,\psi_2,\ldots,\psi_{m_h}\}$ denotes a basis for V_h , with dimension $m_h \to \infty$ as $h \to 0^+$, (ii) $\bigcup_h V_h$ is dense in V . A Faedo-Galerkin approximation in V_h of problem (7.5) is defined as an absolutely continuous function U^h from $[0,T] \to V_h$, i.e., $U^h \in C_A([0,T];V_h)$, solution of the system

$$\left.\begin{array}{l} \langle \dot{U}^h(t),\psi_k \rangle + \langle A(U^h(t)),\psi_k \rangle = \langle f(t),\psi_k \rangle \qquad k = 1,2,\ldots,m_h \\ U^h(0) = U_o^h \end{array}\right\} \tag{10.1}$$

for a.e. $t \in [0,T]$ and where $U_o^h \to u_o$ strongly in $L^2(\Omega)$ as $h \to 0^+$. We observe that if U^h is solution of (10.1), then its time derivative $\dot{U}^h$ belongs to $L^{p'}(0,T;V_h)$ but not necessarily to $\mathcal{V}' = L^{p'}(0,T;W^{-1,p'}(\Omega))$ $(\subset \bigcup_h L^{p'}(0,T;V_h))$.

We next establish the solvability of problem (10.1).

Theorem 10.1. For each $h \in (0,1]$, the Faedo-Galerkin approximation problem (10.1) possesses a unique solution $U^h \in C_A([0,T];V_h)$ continuous with respect to U_o^h.

Proof. The local existence of solutions to (10.1) in $C_A([0,t_h];V_h)$, $t_h > 0$, is implied by the pseudomonotonicity property of A (cf. Remark 7.1). Indeed, $f \in \mathcal{V}$ and A is necessarily bounded and demicontinuous from $V \to V'$ and these are sufficient conditions for the vector field $\underset{\sim}{F}(t,\underset{\sim}{U}) = (\langle f(t),\psi_k\rangle - \langle A(U(t)),\psi_k\rangle)$ from $D = [0,T] \times \mathbb{R}^{m_h} \to \mathbb{R}^{m_h}$ to satisfy the Carathéodory conditions in D. Here $\underset{\sim}{U} \in \mathbb{R}^{m_h}$ denotes the coordinate vector of $U \in V_h$ with respect to the reciprocal basis of V_h.

The uniqueness and continuous dependence on the initial data of local solutions to problem (10.1) follows from the condition [7]: for each compact set $w \subset D$, there is a function $g_w \in L^1(0,T)$ such that

$$|\underset{\sim}{F}(t,\underset{\sim}{U})-\underset{\sim}{F}(t,\underset{\sim}{W})| \leq g_w(t)|\underset{\sim}{U}-\underset{\sim}{W}| \;, \quad (t,\underset{\sim}{U}),(t,\underset{\sim}{W}) \in w \tag{10.2}$$

which is satisfied because A is locally Lipschitz continuous from $V \to V'$ (cf. Remark 7.1).

It remains to be proved that the interval of existence $[0,t_h] = [0,T]$. This is a consequence of the coercivity of A from $V \to V'$, as follows from part (1) of the proof of Theorem 10.2 given below. □

We now proceed to analyze the convergence of the Faedo-Galerkin approximation process.

Theorem 10.2. From the sequence of Faedo-Galerkin approximate solutions defined uniquely by (10.1), there is a subsequence, also denoted $\{U^h\}_{0<h\leq 1}$, and

there exist functions $u \in \mathcal{W}$ and $\mathcal{X} \in \mathcal{V}'$ such that, as $h \to 0^+$,

$$\left.\begin{array}{lll} U^h \rightharpoonup u & \text{weakly in } \mathcal{V} \\ U^h \rightharpoonup u & \text{weakly* in } L^\infty(0,T;L^2(\Omega)) \\ A(U^h) \rightharpoonup \mathcal{X} & \text{weakly in } \mathcal{V}' \\ U^h(T) \rightharpoonup u(T) & \text{weakly in } L^2(\Omega) \end{array}\right\} \tag{10.3}$$

and

$$\left.\begin{array}{l} \left[\dfrac{\partial u}{\partial t}, v\right] + [\mathcal{X}, v] = [f, v] , \quad \forall \; v \in \mathcal{V} \\ u(0) = u_o \end{array}\right\} \tag{10.4}$$

Moreover, the limit function u is a solution of problem (7.5) (i.e., $\mathcal{X} = A(u)$) provided one of the following conditions is satisfied:

i) $\dot{U}^h \in \mathcal{V}'$, $0 < h \leq 1$, and $\{|||\dot{U}^h|||_*\}_{0<h\leq 1}$ is bounded (10.5)

ii) $A: \mathcal{V} \to \mathcal{V}'$ of (7.6) is $\mathcal{V}$-pseudomonotone (10.6)

<u>Proof.</u> We follow the usual pseudomonotone method: (1) a priori bounds, (2) passage to the limit and (3) the pseudomonotonicity argument.

1) From the proof of the coercivity property of A , (7.11), it is apparent that A is also coercive from $V \to V'$:

$$\langle A(v), v\rangle \geq a_o|\nabla v|^2 + \gamma_1 \|v\|^p - \gamma_2 , \quad \forall \; v \in V \tag{10.7}$$

Hence, by integrating equation (10.1) with respect to time from 0 to $\tau \in [0,T]$ and using formula (2.9) and (10.7), we obtain

$$\begin{aligned} \frac{1}{2}|U^h(\tau)|^2 - \frac{1}{2}|U_o^h|^2 + \gamma_1 \int_0^\tau \|U^h(t)\|^p \, dt - \gamma_2 \tau \\ \leq \int_0^\tau \langle f(t), U^h(t)\rangle \, dt \\ \leq \frac{1}{p' b^{p'}} \int_0^\tau \|f(t)\|_*^{p'} \, dt + \frac{b^p}{p} \int_0^\tau \|U^h(t)\|^p \, dt \end{aligned}$$

Then, by choosing $b > 0$ such that $\gamma_1 - b^p/p > 0$, it follows that the sequence $\{U^h\}_{0<h\leq 1}$ is bounded in $\mathcal{V}$ and in $L^\infty(0,T;L^2(\Omega))$.

2) With the previous result and the boundedness of A from $\mathcal{V} \to \mathcal{V}'$ given

by (7.8), the validity of (10.3) follows via weak compactness arguments and, then, upon the passage to the limit in equation (10.1), (10.4) is easily concluded. All of this proceeds as in the case of monotone parabolic problems; cf. [11, Chap. 2].

3) It remains to be proved that, if either (10.5) or (10.6) holds, then

$$[X,v] = [A(u),v] \ , \quad \forall \quad v \in V \tag{10.8}$$

From (10.1), (10.3) and (10.4), we see that

$$\begin{aligned}\lim_{h\downarrow 0} \{[\dot{U}^h,U^h] + [A(U^h),U^h]\} &= \lim_{h\downarrow 0} [f,U^h] = [f,u] \\ &= [\dot{u},u] + [X,u] \\ &= \lim_{h\downarrow 0} \{[\dot{u},U^h] + [A(U^h),u]\}\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{h\downarrow 0} [A(U^h),U^h-u] &= -\lim_{h\downarrow 0} [\dot{U}^h-\dot{u},U^h] = -\frac{1}{2}\lim_{h\downarrow 0} |U^h(T)-u(T)|^2 \\ &\le 0\end{aligned} \tag{10.9}$$

Now, by identical arguments to those given in the proof of (5.5), (10.8) follows from (10.9) and $(10.3)_1$ when assuming either (10.5), and using the W-pseudomonotonicity property of $A: V \to V'$ (cf. Theorem 7.3), or (10.6). This completes the proof of the theorem.

We next show that condition (10.5) is also sufficient for the strong convergence of the approximation process.

<u>Theorem 10.3.</u> Suppose the condition (10.5) holds with bound $\mu' > 0$. Then the subsequence $\{U^h\}_{0<h\le 1}$ of Faedo-Galerkin approximate solutions converging weakly to a solution $u \in W$ of problem (7.5), in the sense of Theorem 10.2, is such that, as $h \to 0^+$

$$\left.\begin{aligned} &U^h \to u \quad \text{strongly in} \quad L^\infty(0,T;L^2(\Omega)) \\ &U^h \to u \quad \text{strongly in} \quad V \end{aligned}\right\} \tag{10.10}$$

In fact, the following approximation error estimates hold $\forall \ Z \in L^p(0,T;V_h)$:

$$|u(\tau)-U^h(\tau)| \le |u_o-U_o^h| + \tilde{K}_1(T,\mu)\,\|u-U^h\|^{p'/2}_{L^p(Q)}$$

$$+ \tilde{K}_2(T,\mu,\mu')\,|||u-Z|||^{1/2} \quad \forall \quad \tau \in [0,T] \tag{10.11}$$

$$|||u-U^h||| \le \tilde{K}_3|u_o-U_o^h|^{2/p} + \tilde{K}_4(T,\mu)\,\|u-U^h\|^{1/(p-1)}_{L^p(Q)}$$

$$+ \tilde{K}_5(T,\mu,\mu')\,|||u-Z|||^{1/p} \tag{10.12}$$

where $\mu > 0$ is a bound for u and $\{U^h\}_{0<h\le\overset{\circ}{1}}$ in V .

<u>Proof.</u> By using formual (2.9) and the Garding-type inequality (7.16) in $L^p(0,\tau;W_o^{1,p}(\Omega))$, $\tau \in [0,T]$, it follows that

$$\int_0^\tau \langle \dot{u}(t)-\dot{U}^h(t) + A(u(t)) - A(U^h(t)),u(t)-U^h(t)\rangle\, dt$$

$$\ge \frac{1}{2}\,|u(\tau)-U^h(\tau)|^2 - \frac{1}{2}\,|u_o-U_o^h|^2 + \alpha_1\int_0^\tau \|u(t)-U^h(t)\|^p dt$$

$$- \alpha_2(\mu)\left\{\int_0^\tau \|u(t)-U^h(t)\|^p_{L^p(\Omega)} dt\right\}^{p'/p} \tag{10.13}$$

and, from equations (7.5) and (10.1), the following orthogonality condition holds:

$$\int_0^\tau \langle \dot{u}(t)-\dot{U}^h(t) + A(u(t)) - A(U^h(t),Z(t)\rangle\, dt = 0$$

$$\forall \quad Z \in L^p(,T;V_h) \tag{10.14}$$

Hence, introducing (10.14) into (10.13) and using the local Lipschitz continuity property (7.7), we obtain

$$\frac{1}{2}\,|u(\tau)-U^h(\tau)|^2 + \alpha_1\int_0^\tau \|u(t)-U^h(t)\|^p dt$$

$$\le \frac{1}{2}\,|u_o-U_o^h|^2 + \alpha_2(\mu)\,\|u-U^h\|^{p'}_{L^p(Q)}$$

$$+ \left\{C(\mu)\,|||u-U^h||| + |||\dot{u}-\dot{U}^h|||_*\right\}|||u-Z|||$$

$$\forall \quad \tau \in [0,T] \ , \quad \forall \quad Z \in L^p(0,T;V_h) \tag{10.15}$$

Therefore, the approximation error estimates (10.11) and (10.12) are implied by (10.15). Note that the strong convergence of $U^h \to u$ in $L^p(Q)$ is a consequence of $(10.3)_1$, assumption (10.5) and the compact embedding of W into $L^p(Q)$

(cf. Theorem 6.1). $\square$

The Potential Case. We conclude this section by showing that if the bounded, coercive, locally Lipschitz continuous, Gårding-type operator A of (7.6), is potential in the sense

CIII. A is the gradient of some Gateaux differentiable functional $J: V \to \mathbb{R}$, for which there is a constant $\tilde{\gamma} > 0$ such that

$$J(v) \geq \tilde{\gamma} \| v \|^p , \quad \forall \; v \in V \tag{10.16}$$

then, for data

$$(f, u_o) \in L^2(Q) \times V \tag{10.17}$$

$$U_o^h \to u_o \quad \text{strongly in } V \tag{10.18}$$

the Faedo-Galerkin sequence of approximations defined uniquely by (10.1) is such that

$$\left.\begin{array}{ll} \{U^h\}_{0<h\leq 1} & \text{is bounded in } L^\infty(0,T;V) \\ \{\dot{U}^h\}_{0<h\leq 1} & \text{is bounded in } L^2(Q) \end{array}\right\} \tag{10.19}$$

Since $L^2(Q) \hookrightarrow \mathcal{V}'$, property $(10.19)_2$ is stronger than (10.5) and, consequently, the results of Theorems 10.2 and 10.3 are true in this potential case.

We now prove this result and establish the corresponding regularity of limit functions.

<u>Theorem 10.4.</u> Let the operator A of (7.6) satisfy condition (CIII) and consider problems (7.5) and (10.1) with data (10.17), (10.18). Then the Faedo-Galerkin sequence of approximate solutions $\{U^h\}_{0<h\leq 1}$ is bounded in the sense of (10.19). Furthermore, there is a subsequence of approximations, also denoted $\{U^h\}_{0<h\leq 1}$, converging strongly to a solution $u \in \mathcal{W}$ of problem (7.5) in the sense of (10.10), such that, as $h \to 0^+$,

$$\left.\begin{array}{lll} U^h \rightharpoonup u & \text{weakly* in } L^\infty(0,T;V) \\ \dot{U}^h \rightharpoonup \dot{u} & \text{weakly in } L^2(Q) \end{array}\right\} \tag{10.20}$$

<u>Proof.</u> Let $\{U^h\}_{0<h\leq 1}$ be the Faedo-Galerkin sequence defined uniquely by (10.1), (10.17), (10.18), which approximates problem (7.5) with data (10.17), and suppose that condition (CIII) holds. Then, by replacing ψ_k by $\dot{U}^h$ in equation (10.1), integrating with respect to time from 0 to $\tau \in [0,T]$ and, then, observing that $dJ(U^h(t))/dt = \langle A(U^h(t)), \dot{U}^h(t)\rangle$ and $(f(t),\dot{U}^h(t)) \leq 1/2|f(t)|^2 + 1/2|\dot{U}^h(t)|^2$ for a.e. $t \in (0,T)$, we obtain

$$\frac{1}{2}\int_0^\tau |\dot{U}^h(t)|^2 dt + \tilde{\gamma}\| U^h(\tau)\|^p \leq J(U_o^h) + \frac{1}{2}\int_0^\tau |f(t)|^2 dt \qquad \forall\ \tau \in [0,T] \tag{10.21}$$

But, from the boundedness of A as a map from $V \to V'$ (cf. Remark 7.1),

$$J(U_o^h) = J_o + \int_0^1 \langle A(sU_o^h), U_o^h\rangle ds < J_o + \int_0^1 \| A(sU_o^h)\|_* ds \| U_o^h\| \leq \text{const.}$$

Therefore, (10.19) is true.

Next observe that from Theorem 10.3, there is a subsequence of approximations $\{U^h\}_{0<h\leq 1}$ that converges strongly to a solution u of problem (7.5) in $\mathcal{V} \cap L^\infty(0,T;L^2(\Omega))$. Hence, $U^h \rightharpoonup u$ weakly in $\mathcal{V}$ ($\hookrightarrow L^1(0,T;V)$ densely) and this together with $(10.19)_1$ is equivalent to $(10.20)_1$. Also $\{U^h\}_{0<h\leq 1}$ is bounded in $\mathcal{U}$ ($\hookrightarrow \mathcal{V}$ densely) and this with $U^h \rightharpoonup u$ weakly in $\mathcal{V}$ is necessary and sufficient for $U^h \rightharpoonup u$ weakly in $\mathcal{U}$ (cf. [17, Sec. V.1]). Then $(10.20)_2$ necessarily holds and this completes the proof of the theorem. □

CONCLUSIONS

For the nonlinear evolution problems considered here, we have shown that coerciveness and $\mathcal{W}$-pseudomonotonicity of $A: \mathcal{V} \to \mathcal{V}'$ guarantees the existence of solutions of (3.1) whereas condition (8.1) implies uniqueness of solutions. The elliptic regularization ideas discussed in Section 4 provide a general framework for Galerkin approximations of $\mathcal{W}$-pseudomonotone problems. We have established criteria for the existence and weak convergence of such approximations in Sections 4, 5 and 9 and strong convergence whenever a Gårding-type inequality of the type in (7.16) holds. More generally, our approximation results in Section 9 apply to operators satisfying inequalities of Gårding-type. In particular, the existence and weak convergence of such approximations were proved in Section 4, 5 a d 9 and, from the analysis of Section 9, it follows that their convergence is strong provided A satisfies a nonlinear Gårding-type inequality of the form

$$[A(v)-A(w), v-w] \geq \alpha_1 \| v-w \|_{\mathcal{V}} - H(\mu, \| v-w \|_{L^p(0,T;X)})$$

$$\forall \quad v,w \in B_\mu(0) \subset \mathcal{V}$$

where $\alpha_1 > 0$, and X is a Banach space continuously embedded in H and in which V is compactly embedded. Also, if in addition, $A: \mathcal{V} \to \mathcal{V}'$ is locally Lipschitz continuous, we have shown that error estimates for Galerkin approximations can be derived.

The Faedo-Galerkin method was considered as an alternative method for constructing approximate solutions. In these cases, coercivity, boundedness and demicontinuity of A from $V \to V'$ are sufficient conditions for existence, and local Lipschitz continuity from $V \to V'$ is a sufficient condition for uniqueness. As we have seen, the convergence of this method is a conditional property in the case that A be non-monotone. In general, we may conclude the following convergence results: Let A satisfy the existence conditions for the abstract problem and its Faedo-Galerkin approximation: the conditions discussed in Sections 3 and 10. Then the Faedo-Galerkin method is weakly convergent if (i) the sequence of time derivatives of the approximate solutions is bounded in $\mathcal{V}'$, or if (ii) $A: \mathcal{V} \to \mathcal{V}'$ is ($\mathcal{V}$-) pseudomonotone. The convergence of the method is strong if (iii) condition (i) holds and A satisfies a nonlinear Gårding inequality of the type given above. Furthermore, in the case in which condition (iii) is satisfied, error estimates are derivable which are compatible with the interpolation theory of finite-elements in Sobolev spaces [12], [3].

A fundamental convergence condition for the Faedo-Galerkin method when applied to $\mathcal{W}$-pseudomonotone parabolic problems is that the sequence time derivatives of the approximate solutions be bounded in $\mathcal{V}'$. We have shown that this condition is satisfied whenever A is, in addition, continuous and potential from $V \to V'$, its potential is coercive, and the data $(f,u_o) \in \mathcal{H} \times V$. In this potential case, the convergence condition holds in $\mathcal{H} \hookrightarrow \mathcal{V}'$. Furthermore, the approximate solutions form a sequence bounded in $L^{\infty}(0,T;V) \hookrightarrow \mathcal{V}$. Then the regularity in time result "$(u,\partial u/\partial t) \in L^{\infty}(0,T;V) \times \mathcal{H}$" holds for the exact solutions of the problem.

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References

1. ADAMS, R.A., Sobolev Spaces, Acad. Press, N.Y., 1975.
2. AUBIN, J.P., "Un Théorème de Compacité", C.R. Acad. Sci. Paris, 256, 1963, pp. 5042-5044.
3. CIARLET, P.G., The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, N.Y., Oxford, 1978.
4. COLEMAN, B.D., MIZEL, V.J., "Thermodynamics and Departures from Fourier's Law of Heat Conduction", Arch. Rational Mech. Anal., 13, 1963, pp. 245-261.
5. DUBINSKII, Yu, A., "Quasilinear Elliptic and Parabolic Equations of Arbitrary Order", Russian Math. Surveys, 23, 1968, pp. 45-92.
6. FITZGIBBON (III), W.E., WALKER, H.F., Nonlinear Diffusion, Pitman, London, San Francisco, Melbourne, 1977.
7. HALE, J.K., Ordinary Differential Equations, Wiley-Interscience, N.Y., 1969.
8. LIONS, J.L., "Sur les Espaces d' Interpolation; Dualité", Math. Scand., 9, 1961, pp. 147-177.
9. _____ "Equations Différentielles Opérationnelles dan les Espaces de Hilbert", C.I.M.E., Roma, Cremonese, 1963, 71 pages.
10. _____ "Sur Certaines Equations Paraboliques Non Linéaires", Bull. Soc. Math. France, 93, 1965, pp. 155-175.
11. _____ Quelques Méthodes de Resolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.
12. ODEN, J.T., REDDY, J.N., An Introduction to the Mathematical Theory of Finite Elements, Wiley-Interscience, N.Y., 1976.
13. ODEN, J.T., "Existence Theorems for a Class of Problems in Nonlinear Elasticity", J. Math. Anal. Appl., (to appear).
14. OLECH, C., OPIAL, Z., "Sur Une Inégalite Differentielle", Ann. Polon. Math., 7, 1960, pp. 247-254.
15. ODEN, J.T., REDDY, C.T. and KIKUCHI,N., "Qualitative Analysis and Finite Element Approximation of a Class of Nonmonotone Nonlinear Dirichlet Problems", TICOM Rept. 78-15, The University of Texas, Austin, 1978.
16. TSUTSUMI, M., "Existence and Nonexistence of Global Solutions for Nonlinear Parabolic Equations", Publ. RIMS, Kyoto Univ., 8, 1972/73, pp. 211-229.
17. YOSIDA, K., Functional Analysis, 4th Ed., Springer Verlag, N.Y., Heidelberg, Berlin, 1974.

APPENDIX D

A Pseudo-Parabolic Variational Inequality and Stefan Problem

1. Introduction.

Let A and B be maximal monotone operators and let C be a non-empty closed convex set in the real Hilbert space H. We shall give existence and uniqueness results for evolution inequalities (formally) of the form

$$(1.1.a) \qquad u(t) \in C: \left(\frac{d}{dt}(Au(t)) + Bu(t) - f(t),\ v - u(t)\right)_H \geq 0\ , \qquad v \in C\ ,\quad 0 \leq t \leq T\ ,$$

$$(1.1.b) \qquad (Au(0) - v_0\ ,\ v - u(0))_H \geq 0\ , \quad v \in C\ ,$$

when $f \in L^2(0,T;H)$ and $v_0 \in A(u_0)$ are given. In Section 2 we introduce a new notion of weak solution of (1.1) and verify uniqueness when A is linear self-adjoint and B is strictly-monotone. Existence of a weak solution is proved when A is strongly-monotone, B is a subgradient, and both operators are locally bounded.

Variational inequalities of the form (1.1) are of interest on their own as extensions of corresponding evolution equations of Sobolev type (where $C = H$). Early work on such inequalities is described in [2]; we mention [6] specifically as a source of examples of initial-boundary value problems for the pseudo-parabolic partial differential equation

$$(1.2) \qquad \frac{\partial}{\partial t}(u - a\Delta u) = k\Delta u$$

with $a > 0$, $k > 0$. Such equations arise as models for diffusion, and they provide an interesting alternative to the classical diffusion equation wherein $a = 0$. In Section 3 we recall the two-temperature heat conduction model from [3] and develop a corresponding one-phase Stefan problem for (1.2). Then we show that such a problem leads to the variational inequality (1.1). This development is parallel to that of the classical case $a = 0$ which is described, e.g., in [7]. Existence of a classical solution of a Stefan problem for (1.2) in one dimension was given in [9] by entirely different methods.

2. The Variational Inequality.

We denote by $L^2(0,T;H)$ the Hilbert space of (Bochner) square-integrable functions on the interval $(0,T)$ with values in the Hilbert space H. Let $H^1(0,T;H)$ denote the absolutely continuous H-valued functions v whose derivatives $\frac{dv}{dt}$ belong to $L^2(0,T;H)$. Denote the dual of H by H^* and recall the natural identification $L^2(0,T;H) = L^2(0,T;H)^*$; thus we obtain the (dual) identification $L^2(0,T;H) \hookrightarrow H^1(0,T;H)^*$ by restriction. The derivative $\frac{d}{dt}: H^1(0,T;H) \to L^2(0,T;H)$ is a bounded linear operator which determines the dual operator $L \equiv -(\frac{d}{dt})^*: L^2(0,T;H) \to H^1(0,T;H)^*$ by the formula

$$\langle Lf,v\rangle \equiv -(f,\frac{dv}{dt})_{L^2}, \quad f \in L^2(0,T;H), \quad v \in H^1(0,T;H).$$

The restriction of Lf to H-valued test functions is the (distribution) derivative $\frac{df}{dt}$. Moreover, for $f \in H^1(0,T;H)$ we have

$$\langle Lf,v\rangle = (\frac{df}{dt}, v)_{L^2} + (f(0),v(0))_H - (f(T),v(T))_H, \quad v \in H^1(0,T;H).$$

Thus, we can regard "$Lf + f(T)$" as formally equivalent to the Cauchy operator "$\frac{df}{dt} + f(0)$".

We shall use basic material on maximal monotone operators [1]. Specifically, recall $A \subset H \times H$ is monotone if $[x_j, y_j] \in A$ for $j = 1$ and 2 imply $(x_1-x_2, y_1-y_2)_H \geq 0$, strictly monotone if in addition equality holds only if $x_1 = x_2$, and strongly monotone if there is a $c > 0$ for which $(x_1-x_2, y_1-y_2)_H \geq c\|x_1-x_2\|_H^2$ for all such pairs $[x_j, y_j]$. If $\varphi: H \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous, its subgradient defined by

$$\partial\varphi(x) = \{u \in H: (u,y-x)_H \leq \varphi(y)-\varphi(x) \text{ for all } y \in H\}$$

for $x \in H$ is maximal monotone. More specifically, if C is a non-empty, convex and closed set in H, its <u>indicator function</u>

$$I_C(x) \equiv \begin{cases} 0 & , \quad x \in C \\ +\infty & , \quad x \notin C \end{cases}$$

is proper, convex and lower semicontinuous, and we have $u \in \partial I_C(x)$ if and only if

$$x \in C: \ (u, y-x) \le 0 \quad \text{for all} \quad y \in C \ .$$

We shall be concerned with maximal monotone operators A with domain $D(A) = H$. That is, $A(x)$ is non-empty for every $x \in H$. This is known to be equivalent to A being <u>locally bounded</u>: for each $x \in H$ there is a neighborhood of x on which A is bounded. This does not imply A is bounded in general unless H has finite dimension or A is linear.

Suppose we are given a pair A,B of maximal monotone operators on the Hilbert space H, a closed convex subset C of H, $f \in L^2(0,T;H)$ and $[u_0, v_0] \in A$. The triple $\{u,v,w\}$ is a <u>strong solution</u> of (1.1) if

$$u,v \in H^1(0,T;H); \ w \in L^2(0,T;H) \ ,$$

(2.1.a) $\quad u(t) \in C: \ \left(\frac{dv(t)}{dt} + w(t) - f(t), \ x - u(t)\right)_H \ge 0 \ , \quad x \in C \ ,$

$v(t) \in A(u(t))$ and $w(t) \in B(u(t))$ for a.e. $t \in [0,T]$, and

(2.1.b) $$(v(0) - v_0 \, , \, x - u(0))_H \ge 0 \ , \quad x \in C \ .$$

Note that since u and v are continuous, C is closed in H and A is closed in $H \times H$, it follows that the inclusions $u(t) \in C$ and $v(t) \in A(u(t))$ hold for all $t \in [0,T]$. Also, (2.1) can be restated as

$$\frac{dv(t)}{dt} + w(t) + \partial I_C(u(t)) \ni f(t) \tag{2.2.a}$$

$$v(0) + \partial I_C(u(0)) \ni v_0 \tag{2.2.b}$$

in terms of the indicator function.

We shall relax the requirement that $v \in H^1(0,T;H)$ as follows. Set $K \equiv \{u \in H^1(0,T;H): u(t) \in C,\ 0 \le t \le T\}$. Define a <u>weak solution</u> of (1.1) to be a triple $\{u,v,w\}$ satisfying

$$u \in K \ ; \quad v,w \in L^2(0,T;H) \ ,$$

$$v(t) \in A(u(t)) \ , \ w(t) \in B(u(t)) \ , \quad \text{a.e.} \quad t \in [0,T] \ ,$$

and for some $\xi \in A(u(T))$ we have

$$\langle Lv + w - f, \eta - u\rangle + (\xi, \eta(T) - u(T))_H \ge (v_0, \eta(0) - u(0))_H \ , \qquad \eta \in K \ . \tag{2.3}$$

Note that if $\{u,v,w\}$ is a strong solution then it is a weak solution with $\xi = v(T)$. Moreover we have the following elementary result.

<u>Theorem 1</u>. Let A be continuous, linear, self-adjoint and monotone; let B be strictly monotone. Then the first two components of a weak solution are uniquely determined.

<u>Proof</u>: Let $\{u_j, v_j, w_j\}$ be weak solutions for $j = 1,2$. By our assumptions on A we may assume (after modification on a null set) $v_j = Au_j \in H^1(0,T;H)$ and $\xi_j = A(u_j(T))$. Thus we have

$$\langle LAu_1 + w_1 - f, u_2 - u_1\rangle + (Au_1(T), u_2(T) - u_1(T))_H \ge (v_0, u_2(0) - u_1(0))_H$$

$$\langle LAu_2 + w_2 - f, u_1 - u_2\rangle + (Au_2(T), u_1(T) - u_2(T))_H \ge (v_0, u_1(0) - u_2(0))_H \ .$$

For any $u \in H^1(0,T;H)$ we have

$$\langle LAu,u\rangle = 1/2((Au(0),u(0))_H-(Au(T),u(T))_H) ,$$

so adding the two inequalities and applying this identity with $u = u_1 - u_2$ gives

$$(w_1-w_2 , u_1-u_2)_{L^2(0,T;H)} + 1/2(Au(T),u(T))_H + 1/2(Au(0),u(0))_H \leq 0 .$$

Strict monotoneity of B shows $u_1 = u_2$.

Remark. Without additional assumptions on the set C we should not expect any uniqueness of the third component, w. For example, in the extreme case $C = \{0\}$, (2.3) is vacuous and we need only choose $v,w \in L^2(0,T;H)$ with $v(t) \in A(0)$ and $w(t) \in B(0)$ to obtain a weak solution. On the other hand, if $C = H$ then any weak solution is a strong solution of the equation $\frac{dv}{dt} + w = f$ in $L^2(0,T;H)$ with initial condition $v(0) = v_0$. See [5] for such Cauchy problems.

The primary objective here is the following existence result.

Theorem 2. Let C be a non-empty, closed convex set in Hilbert space H. Let A and B be maximal monotone operators on H such that A is strongly monotone with domain $D(A) = H$, B is a subgradient, $B = \partial\varphi$, with $D(B) = H$ and $\varphi(x) \geq 0$ for all $x \in H$. For $u_0 \in C$, $v_0 \in A(u_0)$, and $f \in L^2(0,T;H)$ given, there is at least one weak solution $\{u,v,w\}$ of (1.1).

Remarks. Since A is strongly monotone we may assume it is of the form $A + I$. Thus we wish to replace (2.3) by

$$(2.4) \quad \langle L(u+v)+w-f,\eta-u\rangle + (u(T)+\xi,\eta(T)-u(T))_H \geq (u_0+v_0,\eta(0)-u(0))_H, \quad \eta \in K .$$

Proof: For each $\varepsilon > 0$ let I_C^ε be the Yoshida approximation of the indicator function I_C. The subdifferential ∂I_C^ε is a maximal monotone Lipschitz continuous function on H and we have $\partial(\varphi + I_C^\varepsilon) = \partial\varphi + \partial I_C^\varepsilon$. From Theorem 1 of [5] we obtain a strong solution of the approximating Cauchy problem

$$\frac{d}{dt}(u_\varepsilon(t) + v_\varepsilon(t)) + w_\varepsilon(t) + \partial I_C^\varepsilon(u_\varepsilon(t)) = f(t) , \tag{2.5}$$

$$v_\varepsilon(t) \in A(u_\varepsilon(t)) , \; w_\varepsilon(t) \in B(u_\varepsilon(t)) , \quad \text{a.e.} \quad t \in [0,T] ,$$

$$u_\varepsilon(0) = u_0 , \quad v_\varepsilon(0) = v_0 \in A(u_0) ,$$

with $u_\varepsilon , v_\varepsilon \in H^1(0,T;H)$ and $w_\varepsilon \in L^2(0,T;H)$. Taking the inner product with $\frac{du_\varepsilon}{dt}$ in (2.5) and using the chain rule [1, Lemma 3.3] give

$$\int_0^t \left|\frac{du_\varepsilon}{dt}\right|_H^2 d\tau + \int_0^t \frac{d}{d\tau}(\varphi(u_\varepsilon) + I_C^\varepsilon(u_\varepsilon))d \;\le \int_0^t (f, \frac{du_\varepsilon}{dt})_H \, d\tau .$$

(We dropped the non-negative term

$$\int_0^t (\frac{dv_\varepsilon}{dt}, \frac{du_\varepsilon}{dt})_H \, d\tau$$

by monotoneity of A.) Therefore

$$\left\|\frac{du_\varepsilon}{dt}\right\|^2_{L^2(0,T;H)} + \varphi(u_\varepsilon(t)) + I_C^\varepsilon(u_\varepsilon(t)) \le \|f\|_{L^2(0,T;H)} \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2(0,T;H)} + \varphi(u_0) + I_C^\varepsilon(u_0) .$$

Recall that

$$I_C^\varepsilon(x) = \frac{1}{2\varepsilon}|x - \mathrm{Proj}_C(x)|_H^2 , \quad x \in H ; \tag{2.6}$$

thus $I_C^\varepsilon(u_\varepsilon(t)) \ge 0$ and $I_C^\varepsilon(u_0) = 0$ since $u_0 \in C$. From $\varphi(x) \ge 0$, $x \in H$, we obtain

$$\sup_{0\le t\le T} |u_\varepsilon(t)|_H + \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2(0,T;H)} \le M \tag{2.7}$$

$$\sup_{0\le t\le T} I_C^\varepsilon(u_\varepsilon(t)) \le M \tag{2.8}$$

where M depends on u_0 and f but not ε. By the Ascoli-Arzela Theorem we may pass to a suitable subnet (indexed again by ε) and obtain

$$u_\varepsilon(t) \to u(t)\ ,\quad \text{strongly in } H\ ,\quad \text{uniformly in } t\ , \tag{2.9}$$

$$\frac{du_\varepsilon}{dt} \to \frac{du}{dt}\ ,\quad \text{weakly in } L^2(0,T;H)\ . \tag{2.10}$$

The limit $u \in H^1(0,T;H)$ is continuous so its range is a compact path in H. Since A and B are locally bounded there is an ε_0 such that for $0 < \varepsilon \le \varepsilon_0$ and $t \in [0,T]$

$$|v_\varepsilon(t)|_H + |w_\varepsilon(t)|_H \le M\ .$$

Thus we may use the maximality of A and B to pass to a subnet (again indexed by ε) for which

$$v_\varepsilon(T) \to \xi \quad \text{weakly in } H\ ,\quad \text{and} \tag{2.11}$$

$$v_\varepsilon \to v\ ,\quad w_\varepsilon \to w \quad \text{weakly in } L^2(0,T;H)\ , \tag{2.12}$$

where $\xi \in A(u(T))$ and $v(t) \in A(u(t))$, $w(t) \in B(u(t))$ for almost every $t \in [0,T]$.

We shall show the triple $\{u,v,w\}$ is a weak solution. From (2.6), (2.8) follows

$$|u_\varepsilon(t)-\mathrm{Proj}_C(u_\varepsilon(t))|_H^2 \le 2\varepsilon M\ ,\qquad t \in [0,T]\ ,$$

so (2.9), (2.10) show $u \in K$. For any $\eta \in K$ we obtain from (2.5)

$$\begin{aligned}\langle L(u_\varepsilon+v_\varepsilon)+w_\varepsilon-f,\eta-u_\varepsilon\rangle &+ (u_\varepsilon(T)+v_\varepsilon(T),\eta(T)-u_\varepsilon(T))_H\\ &= (-\partial I_C^\varepsilon(u_\varepsilon),\eta-u_\varepsilon)_{L^2(0,T;H)} + (u_\varepsilon(0)+v_\varepsilon(0),\eta(0)-u_\varepsilon(0))_H\ .\end{aligned}$$

From the definition of subgradient and the inclusion $\eta \in K$ we obtain

$$(-\partial I_C^\varepsilon(u_\varepsilon), \eta - u_\varepsilon)_{L^2(0,T;H)} \geq \int_0^T (I_C^\varepsilon(u_\varepsilon(t)) - I_C^\varepsilon(\eta(t))dt \geq 0 ,$$

so there follows

$$\langle L(u_\varepsilon + v_\varepsilon) + w_\varepsilon - f, \eta - u_\varepsilon \rangle + (u_\varepsilon(T) + v_\varepsilon(T), \eta(T) - u_\varepsilon(T))_H \geq (u_0 + v_0, \eta(0) - u_\varepsilon(0))_H .$$

Using (2.9)-(2.12) and the weak continuity of L, we take the limit as $\varepsilon \to 0$ in the preceding inequality and obtain (2.4).

Remark. The approximation of (2.4) by (2.5) is an abstract penalty method.

3. A Stefan Problem.

We consider a problem of heat diffusion involving a solid-liquid phase change at a prescribed temperature. One application we have in mind is the melting of ice (initially at temperature zero) suspended in a reservoir or porous medium. The novelty in this treatment is that we assume the heat diffusion is governed by the pair of equations

$$\frac{\partial \theta}{\partial t} = k\Delta\varphi , \qquad \theta = \varphi - a\Delta\varphi .$$

Chen and Gurtin [3] introduced such a model for heat conduction in non-simple materials where the energy, entropy, heat flux and thermodynamic temperature $\theta(x,t)$ depend on the conductive temperature $\varphi(x,t)$ and its first two spatial gradients. Here the heat flux is determined by the conductive temperature and the phase is determined by the thermodynamic temperature. Thus $\theta > 0$ in the region occupied by water and $\theta = 0$ corresponds to the frozen region.

We describe the geometry of the problem. Let the bounded domain G in $\mathbb{R}^n$ be the medium in which the ice/water is suspended and let its boundary ∂G consist of two disjoint pieces, Γ_0 and Γ_1. Set $\Omega = G \times (0,T)$, where $T > 0$, and note that its lateral boundary is $B_0 \cup B_1$, where $B_j = \Gamma_j \times (0,T)$ for $j = 0,1$. The water-region $\Omega_1 = \{(x,t) \in \Omega: \theta(x,t) > 0\}$ is separated from the ice-region $\Omega_0 = \{(x,t) \in \Omega: \theta(x,t) = 0\}$ by an interface S which is the phase boundary. The unit outward normal on $\partial\Omega_1$ is denoted by $\vec{N} = (\vec{N}_x, N_t)$, $\vec{N}_x \in \mathbb{R}^n$. If $V(t)$ is the velocity in $\mathbb{R}^n$ of the interface at time t, then it follows by the chain rule that $V(t)\cdot\vec{N}_x + N_t = 0$ on S. Set $\vec{n} = \vec{N}_x/\|\vec{N}_x\|$, the unit outward normal in $\mathbb{R}^n$ of the lateral boundary of Ω_1. Of course $n = \vec{N}_x$ on B_1, and $\vec{N}_x = 0$ where $t = 0$ or $t = T$.

The problem is formulated as follows. We are given the conductivity $k > 0$, temperature discrepancy $a > 0$, and latent heat $b > 0$, of the material and a constant $h \geq 0$ representing conductivity across the lateral boundary B_1. The initial thermodynamic temperature $\theta_0(x)$, $x \in G$, and applied conductive temperature $g(x,t)$, $(x,t) \in B_1$, are given with $\theta_0 = 0$ on Γ_0, $\theta_0 > 0$ on Γ_1, and $g \geq 0$. The local form of the problem is to find a pair of non-negative functions θ, φ on Ω for which we have

(3.1) $$\frac{\partial\theta}{\partial t} = k\Delta\varphi\ ,\quad \theta = \varphi - a\Delta\varphi \qquad \text{in } \Omega_1$$

(3.2) $$k\frac{\partial\varphi}{\partial n} + bV(t)\cdot\vec{n} = 0 \qquad \text{on } S$$

(3.3) $$k\frac{\partial\varphi}{\partial n} + h(\varphi - g) = 0 \qquad \text{on } \Gamma_1$$

(3.4) $$\varphi = 0 \qquad \text{on } \Gamma_0$$

(3.5) $$\theta(\cdot,0) = \theta_0 \qquad \text{on } G\ .$$

Note that if θ, φ is a solution of (3.1)-(3.5) and $\theta_0 \geq 0$, then

$$(3.6) \qquad (a/k)\frac{\partial \theta}{\partial t} + \theta = \varphi \quad \text{in } \Omega$$

so it follows that $\varphi = 0$ on $\Omega_0 \cup S$. Since $g \geq 0$, the maximum principle for the elliptic equation in (3.1) on the region $G(t) \equiv \{x \in G: (x,t) \in \Omega_1\}$ shows that $\varphi > 0$ in Ω_1 and $\frac{\partial \varphi}{\partial n} < 0$ on S. Thus $N_t < 0$ on S and $G(t)$ is increasing with t.

We shall show that the problem (3.1)-(3.5) leads to a variational inequality of the form (1.1). Letting $H^1(G)$ denote the Sobolev space of functions v in $L^2(G)$ for which all derivatives $\frac{\partial v}{\partial x_j}$, $1 \leq j \leq n$, belong to $L^2(G)$, we define $V = \{v \in H^1(G): v|_{\Gamma_0} = 0\}$. Here $v|_{\Gamma_0}$ is the trace on the boundary of G; see [8,10] for details. Regarding regularity of a solution, we assume $\theta_0 \in V$, $\theta: [0,T] \to V$ is absolutely continuous, $\varphi \in L^1(0,T;V)$, and (c.f. (3.6))

$$(3.7) \qquad \frac{a}{k}\frac{d\theta(t)}{dt} + \theta(t) = \varphi(t) \;, \quad \text{a.e. } t \in [0,T] \;.$$

Define the continuous linear $B: V \to V^*$ by

$$Bu(v) = \int_G k(\vec{\nabla} u \cdot \vec{\nabla} v)dx + \int_{\Gamma_1} h(uv)ds \;, \qquad u,v \in V \;.$$

For a test function $v \in C_0^\infty((0,T),V)$ we obtain

$$\begin{aligned}
\int_0^T B\varphi(t)(v(t))dt &= \int_{\Omega_1} k\,\vec{\nabla}\varphi \cdot \vec{\nabla} v \, dxdt + \int_{B_1} h\,\varphi\, v\, dsdt \\
&= \int_{\Omega_1} (-k\Delta\varphi)v\, dxdt + \int_{\partial\Omega_1} k\; \vec{\nabla}\varphi \cdot \vec{N}_x v\, dsdt + \int_{B_1} h\,\varphi\, v\, dsdt \\
&= -\int_{\Omega} \frac{\partial \theta}{\partial t} v + \int_{B_1} hgv + \int_S bN_t v
\end{aligned}$$

from (3.1)-(3.4). Furthermore we have

$$\int_S N_t v = \int_{\Omega_1} \frac{\partial v}{\partial t} = \int_\Omega H(\theta)\frac{\partial v}{\partial t} = -\frac{\partial H(\theta)}{\partial t}(v)$$

in the sense of V^*-valued distributions, where $H(s) = 1$ for $s > 0$ and $H(s) = 0$ for $s \leq 0$ is the Heaviside function. We can summarize the above calculations as

$$\text{(3.8)} \qquad \frac{d}{dt}(\theta + bH(\theta)) + B\varphi = (hg)_{\Gamma_1} \quad \text{in } L^1(0,T;V^*)$$

where we define

$$(hg)_{\Gamma_1}(t)(v) = \int_{\Gamma_1} hg(s,t)v(s)ds \ , \quad v \in V \ , \quad t \in [0,T] \ .$$

Combining (3.7) and (3.8) we find that the absolutely continuous function $\theta\colon [0,T] \to V$ satisfies

$$\text{(3.9.a)} \qquad \frac{d}{dt}(\theta + (a/k)B(\theta) + bH(\theta)) + B(\theta) = (gh)_{\Gamma_1} \quad \text{in } L^1(0,T;V^*) \ ,$$

$$\text{(3.9.b)} \qquad \theta(0) = \theta_0 \ , \quad \text{and}$$

$$\text{(3.9.c)} \qquad \theta(x,t) \geq 0 \ , \quad \text{a.e. } x \in G \ , \quad t \in [0,T] \ .$$

If we integrate (3.9.a) and set

$$u(t) = \int_0^t \theta(s)ds$$

$$f(t) \quad (I + (a/k)B + bH)\theta_0 - h + \int_0^t (hg)_{\Gamma_1}(s)ds \ ,$$

there follows

$$\frac{d}{dt}(I + (a/k)B)u + Bu - f(t) = b(1-H(\theta)) \ . \tag{3.10}$$

Finally we note that $H(u) = H(\theta)$ since $G(t)$ is increasing in t, hence, $u(1-H(\theta)) = 0$ in Ω.

The preceding computations show that $u \in H^1(0,T;V)$ and it satisfies $u(0) = 0$,

$$\begin{cases} u(t) \geq 0 \quad \text{in} \quad V \ , \\ \frac{d}{dt}(I + (a/k)B)u(t) + Bu(t) \geq f(t) \quad \text{in} \quad V^* \ , \quad \text{and} \\ (\frac{d}{dt}(I + (a/k)B)u(t) + Bu(t) - f(t))(u(t)) = 0, \ \ 0 \leq t \leq T \ . \end{cases} \tag{3.11}$$

Setting $C \equiv \{v \in V: v \geq 0 \ \text{a.e. in} \ G\}$ we see that u is a strong solution of (1.1) with $A = I + (a/k)B$ and $u_0 = v_0 = 0$; c.f. (2.1). Note that we can trivially rephrase the material of Section 2 in the $H - H^*$ duality instead of the $H - H$ pairing through the scalar product.

Theorem 1 asserts uniqueness of a solution of (3.1)-(3.5) under conditions considerably weaker than those leading to (3.11). Theorem 2 establishes existence of a weak solution to (3.11) which possesses certain additional regularity properties. These topics will be developed elsewhere by other methods [4,11].

REFERENCES

1. H. Brezis, Operateurs Maximaux Monotones et Semigroupes de Contraction dans les Espaces d'Hilbert, Math. Studies 5, North Holland/American Elsevier, 1973.

2. R.W. Carroll and R.E. Showalter, Singular and Degenerate Cauchy Problems, vol. 27, Mathematics in Science and Engineering, Academic Press, New York, 1976.

3. P.J. Chen and M.E. Gurtin, On a theory of heat conduction involving two temperatures, Zeit. Angew, Math. Phys., 19(1968), 614-627.

4. E. DiBenedetto, Stefan problem for pseudo-parabolic equations, in preparation.

5. E. DiBenedetto and R.E. Showalter, Implicit degenerate evolution equations and applications, to appear.

6. J. Lagnese, Perturbations in variational inequalities, Jour. Math. Anal. Appl., 55(1976), 302-238.

7. J. L. Lions, Sur Quelques Question d'Analyse de Mechanique et de Control Optimal, Les Presses de l'Universite de Montreal, Montreal, 1976.

8. J.L. Lions and E. Magenes, Problemes aux Limites Non Homogenes et Applications, vol. 1, Dunod, Paris, 1967.

9. W. Rundell, The Stefan problem for a pseudo-heat equation, Indiana Univ. Math. Jour., 27(1976), 739-750.

10. R.E. Showalter, Hilbert Space Methods for Partial Differential Equations, Pitman, London, 1977.

11. R.E. Showalter, A one-phase pseudo-parabolic Stefan problem, in preparation.

12. T.W. Ting, A cooling process according to two-temperature theory of heat conduction, Jour. Math. Anal. Appl., 45(1974), 23-31.